A NOTE ON RATIONAL ARITHMETIC FUNCTIONS OF ORDER (2, 1)

Pentti Haukkanen
Department of Mathematical Sciences, University of Tampere, P.O. Box 607, SF-33101, Tampere, Finland
(Submitted October 1991)

1. INTRODUCTION

In [4], among other things, the connection between specially multiplicative functions and generalized Fibonacci sequences is discussed. In this paper we shall discuss the similar connection that exists between rational arithmetic functions of order (2, 1) (to be defined in section 2) and generalized Fibonacci sequences. The generalized Fibonacci sequence studied in this paper is the sequence $\{w_n(a,b;c,d)\}$ or, briefly, $\{w_n\}$ of complex numbers, which is defined by

$$w_0 = a,\ w_1 = b,\ w_n = cw_{n-1} - dw_{n-2}\ \ (n \geq 2).$$

This sequence has been extensively studied by Horadam (e.g., [2]).

Section 2 motivates the study of rational arithmetic functions of order (2, 1), while section 3 considers the main theme of this paper, namely, the connection between rational arithmetic functions of order (2, 1) and the sequence $\{w_n\}$. Arising from this connection, identities are presented involving the sequences $\{w_n\}$ and $\{u_n\}$, where $u_n = u_n(c,d) = w_n(1,c;c,d)$. The sequence $\{u_n\}$ is particularly important as indicated in [4]. Finally, in section 4, an identity for rational arithmetic functions of order (2, 1) is proven with the aid of the identities of section 3.

For general background on arithmetic functions, reference is made to the books by Paul McCarthy [3] and Sivaramakrishnan [6]. The basic concepts used in this paper are reviewed here.

An arithmetic function f is said to be multiplicative if $f(1) = 1$ and $f(mn) = f(m)f(n)$ whenever $(m,n) = 1$. If $f(1) = 1$ and $f(mn) = f(m)f(n)$ for all m and n, then f is said to be completely multiplicative. An arithmetic function f is said to be quasi-multiplicative if $f(1) \neq 0$ and there exists a complex number q such that $qf(mn) = f(m)f(n)$ whenever $(m,n) = 1$. It follows immediately that $q = f(1)$. If $f(1) \neq 0$ and $f(1)f(mn) = f(m)f(n)$ for all m and n, then f is said to be a completely quasi-multiplicative function. It is clear that each (completely) multiplicative function is (completely) quasi-multiplicative.

For a prime number p, the generating series of a multiplicative arithmetic function f to the base p is defined by

$$f_p(x) = \sum_{n=0}^{\infty} f(p^n)x^n$$

(see [7]). Each multiplicative function is completely determined by its generating series (at all primes p). It is easy to see that generating series can also be used in the context of quasi-multiplicative functions.

The Dirichlet convolution $f * g$ of two arithmetic functions f and g is defined by

$$(f * g)(n) = \sum_{d|n} f(d)g(n/d).$$

It is clear that, for all primes p, $(f*g)_p(x) = f_p(x)g_p(x)$, when f and g are multiplicative.

2. DEFINITION

The arithmetic function β introduced by S. S. Pillai [5] is given by

$$\beta(n) = \sum_{k=1}^{n} (k, n),$$

where (k, n) is the greatest common divisor of k and n. The structure of β is

$$\beta = I * I * e^{-1} = I * I * \mu, \tag{1}$$

where $I(n) = n, e(n) = 1\,(n \geq 1)$, and μ is the classical Möbius function (see [6, p. 8]). The arithmetic function β is an example of a rational arithmetic function of order (2, 1) in the terminology of Vaidynathaswamy [7], who called a multiplicative arithmetic function f a rational arithmetic function of order (r, s) if there exist nonnegative integers r, s and completely multiplicative functions $g_1, \ldots, g_r$, $h_1, \ldots, h_s$ such that

$$f = g_1 * \cdots * g_r * h_1^{-1} * \cdots * h_s^{-1}.$$

Conventionally, the identity function e_0 is a rational arithmetic function of order (0, 0), where $e_0(1) = 1$ and $e_0(n) = 0$ for $n > 1$.

By (1),

$$\sum_{d|n} \beta(d) = (I * I)(n) = n\tau(n),$$

where $\tau(n)$ is the number of positive divisors of n. The function $n\tau(n)$ is a quadratic function [7], that is, a rational arithmetic function of order (2, 0). A quadratic function is also called a specially multiplicative function in the literature (see, e.g., [4]). If g is specially multiplicative and $g = g_1 * g_2$, then

$$g(m)g(n) = \sum_{d|(m,n)} g(mn/d^2)(g_1 g_2)(d) \tag{2}$$

for all m and n, or, equivalently,

$$g(mn) = \sum_{d|(m,n)} g(m/d)g(n/d)\mu(d)(g_1 g_2)(d) \tag{3}$$

for all m and n (see, e.g., [3, Th. 1.12]). Section 3 includes generalizations of these identities in terms of the sequences $\{w_n\}$ and $\{u_n\}$.

A specially multiplicative function also satisfies

$$g(m)(g_1 g_2)(n) = \sum_{d|n} g(n/d)g(mnd)\mu(d)$$

for all m and n (see [1, Prob. 4, p. 139]). Examination of whether a similar identity holds for β shows that

$$\beta(m)n = \sum_{d|n} \tau(n/d)\beta(mnd)\mu(d)/d$$

for all m and n. Section 4 shows that a similar identity holds for all rational arithmetic functions of order (2, 1).

3. CONNECTIONS WITH GENERALIZED FIBONACCI SEQUENCES

Let g be a specially multiplicative function given by $g = g_1 * g_2$, where g_1 and g_2 are completely multiplicative functions, and let h be a completely quasi-multiplicative function. Let f be defined by $f = g * h^{-1}$. Then $f/f(1)$ is a rational arithmetic function of order (2, 1). Note that $1/f(1) = h(1)$. The generating series of f and g to the base p are

$$f_p(x) = \frac{\frac{1}{h(1)} - \frac{h(p)}{h(1)^2}x}{1 - g(p)x + (g_1g_2)(p)x^2}, \text{ and } g_p(x) = \frac{1}{1 - g(p)x + (g_1g_2)(p)x^2}.$$

The generating series of the sequences $\{w_n\}$ and $\{u_n\}$ are

$$w(x) \equiv \sum_{n=0}^{\infty} w_n x^n = \frac{a + (b - ca)x}{1 - cx + dx^2}, \text{ and } u(x) \equiv \sum_{n=0}^{\infty} u_n x^n = \frac{1}{1 - cx + dx^2}.$$

Thus, for each arithmetic function f given by $f = g * h^{-1}$, where g is a specially multiplicative function and h is a completely quasi-multiplicative function, we have

$$\{f(p^n)\} = \{w_n(f(1), f(p); g(p), (g_1g_2)(p))\}, \text{ and } \{g(p^n)\} = \{u_n(g(p), (g_1g_2)(p))\}.$$

Example 1: For all primes p,

$$\{\beta(p^n)\} = \{w_n(1, 2p-1; 2p, p^2)\}, \text{ and } \{(\beta * e)(p^n)\} = \{u_n(2p, p^2)\}.$$

Conversely, for each sequence $\{w_n\}$ with $a \neq 0$, we have

$$\{w_n(a, b; c, d)\} = \{f(p^n)\}, \text{ and } \{u_n(c, d)\} = \{g(p^n)\},$$

where $f = g * h^{-1}$, g being the specially multiplicative function given by $g(p) = c$, $(g_1g_2)(p) = d$, and h being the completely quasi-multiplicative function given by $h(1) = 1/a$, $h(p) = c/a - b/a^2$. Namely, the above generating series gives $1/h(1) = a, -h(p)/h(1)^2 = b - ca$.

Example 2: For all primes p,

$$\{w_n(2, 1; 1, -1)\} = \{L_n\} = \{f(p^n)\}, \text{ and } \{u_n(1, -1)\} = \{F_{n+1}\} = \{g(p^n)\},$$

where $h(1) = \frac{1}{2}, h(p) = \frac{1}{4}, g(p) = 1, (g_1g_2)(p) = -1$, and F_n, L_n are the Fibonacci and Lucas numbers, respectively.

Using the connection that $w_n = (g_1 * g_2 * h^{-1})(p^n)$ and $u_n = (g_1 * g_2)(p^n)$ it can be proved by some calculations that

$$w_{m+n} = u_m w_n - u_{m-1} w_{n-1} d \quad (m, n \geq 1), \tag{4}$$

and

$$u_m w_n = \sum_{i \leq m, n} w_{m+n-2i} d^i \quad (m, n \geq 1). \tag{5}$$

These identities may be considered as generalizations of the classical identities (2) and (3) for specially multiplicative functions.

There also exist identities that involve generalized Ramanujan sums in identities for specially multiplicative functions (see [6, Th. 124]). The following analogous identities are proposed for the sequences $\{w_n\}$ and $\{u_n\}$: Let $\{\alpha_i\}$ be a sequence of complex numbers, and k, q nonnegative integers. Let $\{\mu_n\}$ be the sequence given by $\mu_0 = 1, \mu_1 = -1, \mu_n = 0 \ (n \ge 2)$. Then we have

$$\sum_{i \le m, n} d^i u_{m-i} w_{n-i} \sum_{\substack{j \le i \\ jk \le q}} \alpha_j \mu_{i-j} = \sum_{\substack{i \le m, n \\ ik \le q}} \alpha_i d^i w_{m+n-2i}, \tag{6}$$

$$\sum_{i \le m, n} w_{m+n-2i} d^i \sum_{\substack{j \le i \\ jk \le q}} \alpha_j = \sum_{\substack{i \le m, n \\ ik \le q}} \alpha_i d^i u_{m-i} w_{n-i}. \tag{7}$$

Note that with $q = 0$ and $\alpha_0 = 1$, (6) and (7) reduce to (4) and (5), respectively.

4. AN IDENTITY

This section presents the identity for rational arithmetic functions of order (2, 1) mentioned at the end of section 2. Let f be a rational arithmetic function of order (2, 1) given by

$$f = g * h^{-1} = g_1 * g_2 * h^{-1},$$

where g_1, g_2, and h are completely multiplicative functions. Use is made of the identity (4) written in terms of f. For all primes p and positive integers r and s,

$$f(p^{r+s}) = g(p^r) f(p^s) - g(p^{r-1}) f(p^{s-1})(g_1 g_2)(p). \tag{8}$$

Theorem: If f is a rational arithmetic function of order (2, 1), then

$$f(m)(g_1 g_2)(n) = \sum_{d|n} g(n/d) f(mnd) \mu(d) \tag{9}$$

for all m and n.

Proof: By multiplicativity, it suffices to consider the case in which m and n are prime powers, say, $m = p^a, n = p^b$. If $b = 0$, both sides of (9) reduce to $f(p^a)$. If $a = 0, b = 1$, then (9) is obtained by (8) with $r = s = 1$. Assume that $a = 0, b > 1$, then the right-hand side of (9) is

$$g(p^b) f(p^b) - g(p^{b-1}) f(p^{b+1}).$$

By (8), this can be written as

$$g(p^b)[g(p^{b-1}) f(p) - g(p^{b-2})(g_1 g_2)(p)] - g(p^{b-1})[g(p^b) f(p) - g(p^{b-1})(g_1 g_2)(p)]$$

or, after simplification,

$$(g_1 g_2)(p)[g^2(p^{b-1}) - g(p^{b-2}) g(p^b)].$$

It can be verified that

$$g^2(p^{b-1}) - g(p^{b-2}) g(p^b) = (g_1 g_2)(p^{b-1})$$

(see [6, Lemma, p. 287]). Since g_1g_2 is completely multiplicative, the left-hand side of (9) is arrived at. The case $a, b > 0$ could be considered in a similar way. The details are not included here.

Remark: Identity (9) in terms of the sequences $\{w_n\}$ and $\{u_n\}$ is:

$$w_n d = u_n w_{m+n} - u_{n-1} w_{m+n+1} \quad (m \geq 0, n \geq 1).$$

ACKNOWLEDGMENT

The author wishes to thank the referee for valuable comments.

REFERENCES

1. T. M. Apostol. *Modular Functions and Dirichlet Series in Number Theory.* New York: Springer-Verlag, 1976.
2. A. F. Horadam. "Basic Properties of a Certain Generalized Sequence of Numbers." *Fibonacci Quarterly* **3.2** (1965):161-77.
3. P. J. McCarthy. *Introduction to Arithmetical Functions.* New York: Springer-Verlag, 1986.
4. P. J. McCarthy & R. Sivaramakrishnan. "Generalized Fibonacci Sequences via Arithmetical Functions." *Fibonacci Quarterly* **28.4** (1990):363-70.
5. S. S. Pillai. "On an Arithmetic Function." *J. Annamalai Univ.* **2** (1937):243-48.
6. R. Sivaramakrishnan. *Classical Theory of Arithmetic Functions.* New York: Marcel Dekker, 1989.
7. R. Vaidyanathaswamy. "The Theory of Multiplicative Arithmetic Functions." *Trans. Amer. Math. Soc.* **33** (1931):579-662.

AMS numbers: 11A25; 11B37