# A FIBONACCI POLYNOMIAL SEQUENCE DEFINED BY MULTIDIMENSIONAL CONTINUED FRACTIONS; AND HIGHER-ORDER GOLDEN RATIOS

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## INTRODUCTION

A sequence of polynomials is a *Fibonacci Sequence* if it satisfies the recursion:

$$f_{n+2}(x) = x \cdot f_{n+1}(x) + f_n(x) \quad \text{for } n \geq 0. \tag{1}$$

Two well-known Fibonacci sequences are the *Fibonacci Polynomials*, $\{F_n(x)\}$, defined using (1) with $F_1(x) = 1$ and $F_2(x) = x$, and the *Lucas Polynommials*, $\{L_n(x)\}$, defined using (1) with $L_1(x) = 2$ and $L_2(x) = x$ ([1], [3], [4], [11], [12], [22], [23]). In addition to being Fibonacci sequences, these polynomials produce Fibonacci and Lucas numbers, respectively, when evaluated at $+1$.

Here we examine a sequence of polynomials $\{G_n(x)\}$ originating from multidimensional continued fractions with all one's. The golden ratio is a root of the quadratic polynomial in this sequence; hence, there is justification to consider the roots of the other polynomials in this sequence to be higher-order golden ratios. Surprisingly, these polynomials also form a Fibonacci sequence, and Fibonacci and Lucas numbers result when evaluated at $+1$ and $-1$, respectively.

It turns out that the Fibonacci and Lucas polynomials, as well as this new sequence are examples of a larger class of Fibonacci polynomial sequences. We develop an explicit formula for this class and show specifically how the Fibonacci numbers are involved when evaluated at $\pm 1$.

## 1. DEFINITION OF THE GOLDEN POLYNOMIALS $\{G_n(x)\}$

The continued fraction

$$1+\cfrac{1}{1+\cfrac{1}{1+\cdots}} \tag{2}$$

satisfies the equation

$$x = 1 + \frac{1}{x}$$

which is readily converted to the polynomial equation $x^2 - x - 1 = 0$. We define $G_2(x)$ to be this second-degree polynomial, and denote its positive root as $g_2$. This root is the value of the continued fraction in (2), namely, the golden ratio

$$g_2 = \frac{1+\sqrt{5}}{2}.$$

Now consider a continued fraction of the form

$$1+\cfrac{1}{\left(1+\cfrac{1}{1+\cdots}\right)+\cfrac{1}{\left(1+\cfrac{1}{1+\cdots}\right)}}. \tag{3}$$

Whereas the sequence of denominators in the continued fraction in (2) could be written in a list, the denominators of (3) would require a binary tree of all 1's. This continued fraction can be written as

$$x=1+\cfrac{1}{x+\cfrac{1}{x}}$$

or as the polynomial equation $G_3(x)=x^3-x^2-1=0$, which has the value of (3) as a solution. Analogously, we designate this single positive root by $g_3$ as an indication that it is a root of the third-degree polynomial $G_3(x)$.

This process can be extended. Consider the family of recursive equations of the form

$$x=1+\underbrace{\cfrac{1}{x+\cfrac{1}{x+\cfrac{1}{x+\cfrac{1}{\cdots+\cfrac{1}{x}}}}}}_{n-1\ x\text{'s}} \tag{4}$$

These equations represent multidimensional continued fractions of all 1's that have n – 1 branches at each level. For each $n$, this equation can be transformed into an $n^{\text{th}}$ degree polynomial equation $G_n(x)=0$. (For $n$ = 1, there are no $x$'s on the right side, so it is natural to define $G_1(x)=x-1$.)

In this way, we get a sequence of functions $\{G_n(x)\}$. Since each function $G_n(x)$ has a positive maximal root $g_n$ [17], we also obtain a sequence of positive numbers $\{g_n\}$. In Section 5, we will see that there is justification to consider these roots to be higher-order golden ratios. Because of this, we will refer to these polynomials $\{G_n(x)\}$ as the "Golden Polynomials."

For example, we can write the coefficients for some of these polynomials as shown in Figure 1. Whereas the sum of the coefficients in the $n^{\text{th}}$ row of Pascal's triangle is $2^n$, the sum shown here is $-F_{n-1}$ (proven in Corollary 2.4 below). For other approaches to the generalization of the continued fraction algorithm, the reader is referred to Bernstein [2] and Szerkeres [21].

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $n=1$ | 1 | −1 | | | | | | | | | |
| $n=2$ | 1 | −1 | −1 | | | | | | | | |
| $n=3$ | 1 | −1 | 0 | −1 | | | | | | | |
| $n=4$ | 1 | −1 | 1 | −2 | −1 | | | | | | |
| $n=5$ | 1 | −1 | 2 | −3 | −1 | −1 | | | | | |
| $n=6$ | 1 | −1 | 3 | −4 | 0 | −3 | −1 | | | | |
| $n=7$ | 1 | −1 | 4 | −5 | 2 | −6 | −2 | −1 | | | |
| $n=8$ | 1 | −1 | 5 | −6 | 5 | −10 | −2 | −4 | −1 | | |
| $n=9$ | 1 | −1 | 6 | −7 | 9 | −15 | 0 | −10 | −3 | −1 | |
| $n=10$ | 1 | −1 | 7 | −8 | 14 | −21 | 5 | −20 | −5 | −5 | −1 |

**FIGURE 1**

## 2. FIBONACCI POLYNOMIAL SEQUENCES

A convenient way to express the Fibonacci recursion in (1) is to define the functional

$$\Phi(f, g) = x \cdot f(x) + g(x).$$

Similarly, we can represent a Fibonacci sequence generated by this functional by

$$\Phi\{f_1, f_0\} = \{f_n \mid f_{n+2} = \Phi(f_{n+1}, f_n) \text{ for } n \geq 0\}.$$

This notation emphasizes that the entire sequence depends only on the two seed functions.

All Fibonacci sequences can be represented in this way. For example,

$$\Phi\{x, 1\} = \{1, x, x^2+1, \ldots\} = \{F_n(x)\} = \text{the Fibonacci Polynomials;}$$
$$\Phi\{x, 2\} = \{2, x, x^2+2, \ldots\} = \{L_n(x)\} = \text{the Lucas Polynomials.}$$

It is clear that there are many such sequences, and we let $\mathcal{F}$ denote the set of all sequences generated in this way. Other approaches to the structure of Fibonacci-type polynomials have been pursued in Horadam [14], Shannon [19], and Dilcher [9].

A number of simple properties are evident.

***Observation 2.1:***

*a.* $\Phi(c \cdot f, c \cdot g) = c \cdot \Phi(f, g)$ for any constant $c$.

*b.* $\Phi(f_1, g_1) + \Phi(f_2, g_2) = \Phi(f_1 + f_2, g_1 + g_2)$.

*c.* $\{f_n\} \in \mathcal{F} \Leftrightarrow \{-f_n\} \in \mathcal{F}$.

*d.* $\{f_n\}, \{g_n\} \in \mathcal{F} \Rightarrow \{f_n + g_n\} \in \mathcal{F}$.

*e.* If $h_n = \Phi(f_n, g_n)$, where $\{f_n\}, \{g_n\} \in \mathcal{F}$, then $\{h_n\} \in \mathcal{F}$.

To show that $\{G_n(x)\}$ is a Fibonacci sequence, we will need the following lemma.

***Lemma 2.2:*** The polynomial numerator and denominator obtained by simplifying the expression

$$\underbrace{x+\cfrac{1}{x+\cfrac{1}{x+\cdots}}}_{n\ x\text{'s}}$$

are $F_{n+1}(x)$ and $F_n(x)$ respectively.

***Proof:*** The lemma is easily verified for $n = 1$ and $n = 2$. Now assume the lemma holds for all $k < n$. We can then write the expression with $n$ $x$'s as follows:

$$\underbrace{\left(x+\cfrac{1}{x+\cfrac{1}{x+\cfrac{1}{\cdots}}}\right)}_{n\ x\text{'s}} = x+\cfrac{1}{\underbrace{\left(x+\cfrac{1}{x+\cfrac{1}{\cdots}}\right)}_{n-1\ x\text{'s}}} = x+\cfrac{1}{\left(\cfrac{F_n(x)}{F_{n-1}(x)}\right)} = \frac{xF_n(x)+F_{n-1}(x)}{F_n(x)} = \frac{F_{n+1}(x)}{F_n(x)}.$$

Noting that the consecutive Fibonacci polynomials share no common factors, and that $\{F_n(x)\} = \Phi\{x, 1\} \in \mathcal{F}$, this completes the proof. □

We now show that the sequence of Golden Polynomials $\{G_n(x)\}$ is a Fibonacci sequence.

***Theorem 2.3:*** $\{G_n(x)\} \in \mathcal{F}$.

***Proof:*** Substituting $F_n(x)$ and $F_{n-1}(x)$ into (4) gives

$$x = 1+\cfrac{1}{\underbrace{\left(x+\cfrac{1}{x+\cfrac{1}{\cdots}}\right)}_{n-1\ x\text{'s}}} = 1+\cfrac{1}{\left(\cfrac{F_n(x)}{F_{n-1}(x)}\right)} = \frac{F_n(x)+F_{n-1}(x)}{F_n(x)}.$$

Simplifying, we have

$$G_n(x) = xF_n(x) - (F_n(x)+F_{n-1}(x)) = 0. \tag{5}$$

By Observations 2.1.c and 2.1.d and Lemma 2.2, we have $\{-(F_n(x)+F_{n-1}(x))\} \in \mathcal{F}$. By Observation 2.1.e, it follows that $\{G_n(x)\} \in \mathcal{F}$. □

The Golden Polynomials are easily seen to be

$$\{G_n(x)\} = \Phi\{x-1, -1\}.$$

During the proof we discovered a relationship between these polynomials $\{G_n(x)\}$ and the Fibonacci Polynomials $\{F_n(x)\}$. Rewriting (5), we have $G_n(x) = (x-1)F_n(x) - F_{n-1}(x)$. Evaluating at $\pm 1$ gives

***Corollary 2.4:***

*a.* $G_n(1) = -F_{n-1}$.

*b.* $G_n(-1) = (-1)^n L_{n-1}$.

This establishes another connection between continued fractions and the Fibonacci and Lucas numbers.

## 3. A SIMPLE GENERALIZATION

A familiar method of generalizing the golden ratio (Coleman [7], Raab [18], Bicknell & Hoggatt [4]) is to define "silver" and various other metallic ratios by forming a rectangle of dimensions 1 by $x$ and removing $c$ unit squares (see Figure 2 below). If the remaining rectangle is similar to the original, then $x$ is called a "generalized golden ratio."

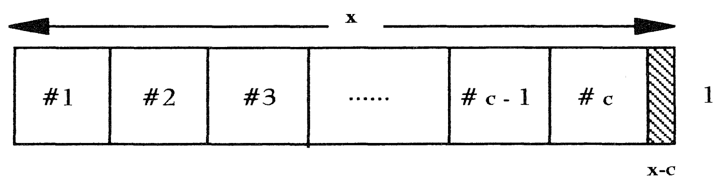


**FIGURE 2**

It is easily demonstrated that these numbers are precisely those that are expressed by continued fractions of period 1. If we then consider multidimensional continued fractions of period 1 and write them recursively as before, we would have a sequence of polynomials corresponding to each positive integer. For example, the cubic continued fraction of period 1, with $c$ as the constant in the denominators, is

$$x = c + \cfrac{1}{\left((c+\cdots) + \cfrac{1}{(c+\cdots)}\right) + \cfrac{1}{\left((c+\cdots)+\cfrac{1}{(c+\cdots)}\right)}} = c + \cfrac{1}{x + \cfrac{1}{x}}.$$

Simplifying gives the third-degree polynomial $H_3(x, c) = x^3 - cx^2 - c$. In this way, we obtain the sequence $\{H_n(x, c)\} = \Phi\{x - c, -c\}$ where the coefficients of $H_n(x, c)$ are the same as those $G_n(x)$ with every other coefficient having an additional factor of $c$. In fact, these polynomials satisfy the relation $H_n(x, c) = (1/2)\{(1+c)G(x) + (1-c)G(-x)]$. Theorem 2.3 is easily extended to these polynomial sequences as well, i.e., $\{H_n(x, c)\} \in \mathscr{F}$.

## 4. AN EXPLICIT FORMULA FOR FIBONACCI SEQUENCES

Consider the Fibonacci sequences of the form $\Phi\{ax + b, c\}$. This class includes both the Fibonacci and Lucas Polynomials as well as the Golden Polynomials $\{G_n(x)\}$ and generalized Golden Polynomials $\{H_n(x; c)\}$. Here we examine an explicit formula for the functions in such a sequence.

We will use the following notational conventions:

**a. Binomial Coefficients:** $C_{n,k}=\binom{n}{k}=\begin{cases}\frac{n!}{k!(n-k)!}, & \text{for } 0\le k\le n,\\ 0, & \text{for } k<0 \text{ or } k>n.\end{cases}$

**b. Greatest Integer Function:** $\lfloor x\rfloor = k$, for the greatest integer function.

**c. Parity Function:** $\delta_k=\begin{cases}1 & \text{if } k \text{ is even,}\\ 0 & \text{if } k \text{ is odd.}\end{cases}$

***Theorem 4.1:*** For $f_n(x)\in\Phi\{ax+b,c\}$,

$$f_n(x)=\sum_{k=0}^{n}R_{n,k}x^{n-k},$$

where

$$R_{n,k}=S_{n,k}\cdot(a\cdot\delta_k+b\cdot(1-\delta_k))+S_{n-1,k-2}\cdot(c\cdot\delta_k)$$

and

$$S_{n,k}=\binom{n-\lfloor\frac{k}{2}\rfloor-1}{\lfloor\frac{k}{2}\rfloor}.$$

***Proof:*** The formula is verified by direct computation for $n=1$ and $n=2$:

$f_1(x)=(a\cdot1+b\cdot0+c\cdot0)x+(a\cdot0+b\cdot1+c\cdot0)=ax+b$, and

$f_2(x)=(a\cdot1+b\cdot0+c\cdot0)x^2+(a\cdot0+b\cdot1+c\cdot0)x+(a\cdot0+b\cdot0+c\cdot1)=ax^2+bx+c.$

Proceeding by induction, we write

$$\begin{aligned}f_{n+2}(x)=xf_{n+1}(x)+f_n(x)&=x\sum_{k=0}^{n+1}R_{n+1,k}x^{n+1-k}+\sum_{k=0}^{n}R_{n,k}x^{n-k}\\&=\sum_{k=0}^{n+1}R_{n+1,k}x^{n+2-k}+\sum_{k=0}^{n}R_{n,k}x^{n-k}\\&=R_{n+1,0}x^{n+2}+R_{n+1,1}x^{n+1}+\sum_{k=0}^{n-1}\left((R_{n+1,k+2}+R_{n,k})x^{n-k}\right)+R_{n,n}.\end{aligned}$$

It now becomes a matter of verifying that the coefficients are correct. The first two terms are:

$$\begin{aligned}R_{n+1,0}&=S_{n+1,0}(a\cdot1+b\cdot1)+0\cdot c(1)=a\cdot S_{n+1,0}\\&=a\cdot C_{n,0}=a\cdot1=a\cdot C_{n+1,0}=a\cdot S_{n+2,0}=R_{n+2,0}\\R_{n+1,1}&=S_{n+1,1}(a\cdot0+b\cdot1)+0\cdot c(0)=b\cdot S_{n+1,1}\\&=b\cdot C_{n,0}=b\cdot1=b\cdot C_{n+1,0}=b\cdot S_{n+2,1}=R_{n+2,1}.\end{aligned}$$

Now consider the constant term

$$R_{n,n}=S_{n,n}(a\cdot\delta_k+b\cdot(1-\delta_k))+S_{n-1,n-2}c\cdot\delta_k.$$

If $n$ is odd, $n = 2m + 1$, and we have

$$R_{n,n} = b \cdot S_{n,n} = b \cdot C_{m,m} = b \cdot 1 = b \cdot C_{m+1,m+1} = b \cdot S_{n+2,n+2} = R_{n+2,n+2}.$$

If $n$ is even, $n = 2m$, and then

$$S_{n,n} = C_{m-1,m} = 0 = C_{m+1,m+2} = S_{n+2,n+2};$$
$$S_{n-1,n-2} = C_{m-1,m-1} = 1 = C_{m+1,m+1} = S_{n+1,n}.$$

Substituting these in, we have

$$\begin{aligned} R_{n,n} &= a \cdot S_{n,n} + c \cdot S_{n-1,n-2} = a \cdot 0 + c \cdot 1 \\ &= a \cdot S_{n+2,n+2} + c \cdot S_{n+1,n} = R_{n+2,n+2}. \end{aligned}$$

All of the other coefficients are of the form:

$$R_{n+1,k} + R_{n,k-2} = [S_{n-1,k} + S_{n,k-2}](a \cdot \delta_k + b \cdot (1-\delta_k)) + [S_{(n+1)-1,k-2} + S_{n-1,(k-2)-2}]c \cdot \delta_k).$$

It will suffice to show that $S_{n+1,k} + S_{n,k-2} = S_{n+2,k}$. Writing $j = [k/2]$, we have

$$S_{n+1,k} + S_{n,k-2} = C_{n-j,j} + C_{n-j,j-1} = C_{n+1-j,j} = S_{n+2,k}$$

by the well-known additive relationship of Pascal's triangle. □

Noting that $\{G_n(x)\} = \Phi\{x-1,-1\}$, we have an explicit formula for the Golden Polynomials.

***Corollary 4.2:*** $G_n(x) = \sum_{k=1}^{n} \left( (S_{n,k} - S_{n-1,k-2})\delta_k - S_{n,k}(1-\delta_k) \right) x^{n-k}$.

We can also make a number of simple observations about this type of sequence.

***Corollary 4.3:*** For each $f_n \in \Phi\{ax+b, c\}$,

***a.*** (the leading coefficient of $f_n(x)$) = (the leading coefficient of $f_1(x)$) = $a$.

***b.*** (the trace of $f_n(x)$) = $(-1)^n \cdot$ (the trace of $f_0(x)$) = $(-1)^n \cdot c$.

***c.*** (the norm of $f_n(x)$) = (the norm of $f_1(x)$) = $b$.

***d.*** $f_{2n}(0) = f_0(0) = c$ and $f_{2n-1}(0) = f_1(0) = b$.

We can now see how Fibonacci numbers are present in all sequences of this type.

***Corollary 4.4:*** For each $f_n \in \Phi\{ax+b, c\}$,

$$f_n(1) = a \cdot F_n + b \cdot F_n + c \cdot F_{n-1} \text{ and } f_n(-1) = (-1)^n (a \cdot F_n - b \cdot F_n + c \cdot F_{n-1}).$$

***Proof:*** Theorem 4.1 can be expressed more conveniently as

$$f_n(x) = a \sum_{\substack{k=0 \\ k \text{ even}}}^{n} S_{n,k} x^{n-k} + b \sum_{\substack{k=0 \\ k \text{ odd}}}^{n} S_{n,k} x^{n-k} + c \sum_{\substack{k=0 \\ k \text{ even}}}^{n} S_{n-1,k-2} x^{n-k}.$$

Evaluating at 1 gives

$$f_n(1) = a \sum_{\substack{k=0 \\ k \text{ even}}}^{n} S_{n,k} + b \sum_{\substack{k=0 \\ k \text{ odd}}}^{n} S_{n,k} + c \sum_{\substack{k=0 \\ k \text{ even}}}^{n} S_{n-1,k-2}. \tag{6}$$

Evaluating at -1 gives

$$f_n(-1) = \begin{cases} a \sum\limits_{\substack{k=0 \\ k \text{ even}}}^{n} S_{n,k} - b \sum\limits_{\substack{k=0 \\ k \text{ odd}}}^{n} S_{n,k} + c \sum\limits_{\substack{k=0 \\ k \text{ even}}}^{n} S_{n-1,k-2} & \text{for } n \text{ even}, \quad (7) \\ -a \sum\limits_{\substack{k=0 \\ k \text{ even}}}^{n} S_{n,k} + b \sum\limits_{\substack{k=0 \\ k \text{ odd}}}^{n} S_{n,k} - c \sum\limits_{\substack{k=0 \\ k \text{ even}}}^{n} S_{n-1,k-2} & \text{for } n \text{ odd}. \quad (8) \end{cases}$$

We simplify these sums using the Fibonacci identity

$$F_{n+1} = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-j}{j},$$

which can be found in [4]. Applying this to the first sum in each of (6)-(8), we have

$$\sum_{\substack{k=0 \\ k \text{ even}}}^{n} S_{n,k} = \sum_{\substack{k=0 \\ k \text{ even}}}^{k \le n} \binom{n - \left[\frac{k}{2}\right] - 1}{\left[\frac{k}{2}\right]} = \sum_{j=0}^{\left[\frac{n-1}{2}\right]} \binom{(n-1)-j}{j} = F_n.$$

Similarly, the second and third sums become

$$\sum_{\substack{k=0 \\ k \text{ odd}}}^{n} S_{n,k} = F_n \quad \text{and} \quad \sum_{\substack{k=0 \\ k \text{ even}}}^{n} S_{n-1,k-2} = F_{n-1}.$$

Substituting these into equations (6)-(8) gives the results. □

## 5. HIGHER-ORDER GOLDEN RATIOS

The applications of the golden ratio to geometry and the Fibonacci numbers are well documented ([5], [15]). Since the root $g_2$ has the value of the golden ratio, it is natural to ask if the other maximal roots $\{g_n\}$ have similar properties. It appears that this is the case. In the four examples that follow, we examine how the $\{g_n\}$ can be considered generalizations of the golden ratio to higher dimensions.

## 5.1 Geometric Properties

Consider a square of side $x$ (labeled "square A" in the diagram below), containing a unit square (square B). Extending the sides of the unit square forms a third square of side $x - 1$ (square C).

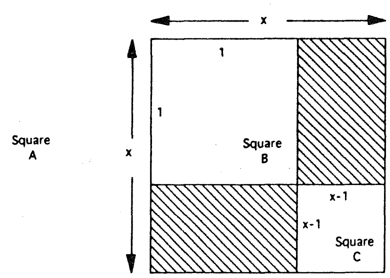


**FIGURE 3**

Note that the ratio of (the side of A) to (the side of B) is equal to the ratio of (the side of B) to (the side of C) only if $x$ is the *golden ratio,* $g_2$. That is $x/1 = 1/(x-1)$. Note also, however, that the ratio of (the area of A) to (the side of B) is equal to the ratio of (the area of B) to (the side of C) only if $x$ is $g_3$. That is $x^2/1 = 1^2(x-1)$.

A golden cuboid is a solid of unit volume having sides in the ratio of $g_2 : 1 : 1/g_2$ (Huntley [15]). It has the property that removing a slice off the top of dimensions $1/g_2 : 1 : 1/g_2$ leaves a smaller solid with the ratio of the volumes being $g_2$. We can analogously define a "platinum cuboid" of dimensions $g_3 : 1 : 1/g_3$. If instead of *removing* a slab of dimensions $1/g_3 : 1 : 1/g_3$, we *add* such a slab, the resulting ratio of volumes is $g_3$ (see Fig. 4).

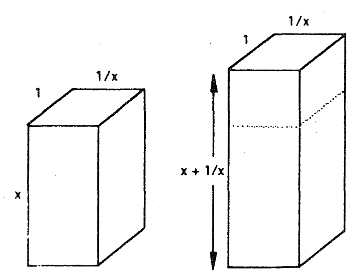


**FIGURE 4**

## 5.2 Continued Fractions and Continued Radicals

By definition, the $\{g_n'\}$ are precisely those numbers that can be expressed by multidimensional continued fractions using all 1's. This is perhaps the strongest argument to consider these numbers as higher-order golden ratios.

It is perhaps worth noting, since 1993 is the 400th anniversary of Vieta's continued radical expression for $\pi$ (Smith [20]), that continued radicals were used extensively in past centuries (Cohen [6] and Shannon [19]). The golden ratio can also be expressed by continued radicals. That is,

$$g_2 = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}} = \sqrt{1+g_2}.$$

Similarly, $g_3$ can be expressed with continued cube roots as

$$g_3 = \sqrt[3]{1+(1+(1+(1+\cdots)^{\frac{2}{3}})^{\frac{2}{3}})^{\frac{2}{3}}} = \sqrt[3]{1+(g_3)^{\frac{2}{3}}}.$$

## 5.3 Rational Sequences

The golden ratio, $g_2$, is the limit of consecutive Fibonacci numbers. This can be expressed as

$$g_2 = \lim_{k\to\infty} \frac{p_k}{q_k} \quad \text{where} \begin{cases} p_1 = q_1 = 1, \\ q_k = p_{k-1}, \\ p_k = p_{k-1} + q_{k-1}. \end{cases}$$

Similarly, there is a rational sequence that converges to $g_3$ defined by

$$g_3 = \lim_{k\to\infty} \frac{p_k}{q_k} \quad \text{where} \begin{cases} p_1 = q_1 = 1, \\ q_k = p_{k-1}^2 + q_{k-1}^2 & = q_{k-1}^2 F_2\left(\frac{p_{k-1}}{q_{k-1}}\right), \\ p_k = p_{k-1}^2 + p_{k-1}q_{k-1} + q_{k-1}^2 & = q_{k-1}^2\left(F_2\left(\frac{p_{k-1}}{q_{k-1}}\right) + F_1\left(\frac{p_{k-1}}{q_{k-1}}\right)\right). \end{cases}$$

Instead of the Fibonacci numbers, the convergents are 1/1, 3/2, 19/13, 797/550, ..., etc. In fact, a rational sequence can be constructed for each $g_n$ using the Fibonacci Polynomials $F_{n-1}(x)$ and $F_{n-2}(x)$. Specifically, for a sequence that converges to $g_{n+1}$, begin with $p_1 = q_1 = 1$, then continue

$$\frac{p_{k+1}}{q_{k+1}} = \frac{q_k^n\left(F_n\left(\frac{p_k}{q_k}\right) + F_{n-1}\left(\frac{p_k}{q_k}\right)\right)}{q_k^n F_n\left(\frac{p_k}{q_k}\right)} = 1 + \frac{F_{n-1}\left(\frac{p_k}{q_k}\right)}{F_n\left(\frac{p_k}{q_k}\right)}.$$

## 5.4 Generated Integer Sequences

The Fibonacci and Lucas numbers are integer sequences generated by the golden ratio and its real conjugate using the Binet forms. In a similar way, we can define the sequence $g_3$ by

$$u_n = \frac{g_3^n + h_3^n + \bar{h}_3^n}{g_3 + h_3 + \bar{h}_3},$$

where $h_3$ and $\bar{h}_3$ are the complex conjugate roots of $G_3$. It can be shown that $\{u_n\}$ is the integer sequence defined by the recursive formula $u_{n+3} = u_{n+2} + u_n$ with initial values $u_0 = 3, u_1 = 1$, and $u_2 = 1$. This gives a "delayed" Fibonacci-type sequence (3), 1, 1, 4, 5, 6, 10, 15, 21, 31, 46, 67, 98, ..., etc. See [8] for additional information on this.

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