SUMS OF UNIT FRACTIONS HAVING LONG CONTINUED FRACTIONS

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Let

$$[x, y, \ldots, z] = x + \cfrac{1}{y + \ddots + \cfrac{1}{z}}$$

be a simple continued fraction. In the representation of an element of $\mathbb{Q} \setminus \mathbb{Z}$ as a simple continued fraction, we normalize by $z \geq 2$. A "unit fraction" $\frac{1}{a} = [0, a]$ for $a \geq 2$ is a very short, simple continued fraction. One may ask if the sum, difference of two unit fractions (with relatively prime denominators) can have arbitrarily many terms in its simple continued fraction. We use the Fibonacci numbers to show that the answer is "yes."

Letting $\frac{1}{F_n} + \frac{1}{F_{n+1}^2} = [0, F_n - 1, \rho_n]$ for $n > 2$, we find $\rho_n \to \phi = \frac{1+\sqrt{5}}{2} = [1, 1, 1, \ldots]$ as $n \to \infty$. Letting $\frac{1}{F_n} - \frac{1}{F_{n+1}^2} = [0, F_n, \sigma_n]$ for $n > 1$, we find that $\sigma_n \to 1 + \phi$ as $n \to \infty$.

The rates of convergence can easily be estimated. For this, instead of using $\frac{1}{a} \pm \frac{1}{b^2}$, it is better to use

$$\frac{1}{a} - \frac{1}{b^2 + a} = \left[0, a, \frac{b^2}{a^2}\right] \text{ and } \frac{1}{a} + \frac{1}{b^2 + a^2 - a} = \left[0, a-1, 1, \frac{b^2}{a^2}\right] \tag{1}$$

where $2 \leq a < b$. Starting with $\frac{F_3}{F_2} = 2 = [2]$ and $\frac{F_4}{F_3} = \frac{3}{2} = [1, 2]$, it is easy to show by induction that

$$\frac{F_{n+1}}{F_n} = [\underbrace{1, \ldots, 1}_{n-2}, 2] \text{ for } n > 1. \tag{2}$$

Lemma: For $n \geq 4$, we have

$$\frac{F_{n+1}^2}{F_n^2} = \left[\underbrace{2, 1, \ldots, 1}_{n-3}, 3, \frac{F_{n-1}}{F_{n-2}}\right]. \tag{3}$$

Proof: Subtracting 2 from both sides of (3) gives, equivalently,

$$\frac{F_n^2}{F_{n+1}^2 - 2F_n^2} = [\underbrace{1, \ldots, 1}_{n-3}, B \text{ where } B = \left[3, \frac{F_{n-1}}{F_{n-2}}\right].$$

But

$$[\underbrace{1, \ldots, 1}_{n-3}, B = \frac{BF_{n-2} + F_{n-3}}{BF_{n-3} + F_{n-4}}.$$

Multiplying the right-hand side by $\frac{F_{n-1}}{F_{n-1}}$ and substituting for B, the numerator turns out to be F_n^2 and the denominator is $F_{n+1}^2 - 2F_n^2$, as it should be.

Letting $a = F_n, b = F_{n+1}$, we have $(a, b^2 + a) = 1$ and $(a, b^2 + a^2 - a) = 1$. Using (3), (2), and (1), we have

$$\frac{1}{F_n} - \frac{1}{F_{n+1}^2 + F_n} = [0, F_n, 2, \underbrace{1, \ldots, 1}_{n-3}, 3, \underbrace{1, \ldots, 1}_{n-4}, 2] \text{ for } n > 3 \tag{4}$$

and

$$\frac{1}{F_n} + \frac{1}{F_{n+1}^2 + F_n^2 - F_n} = [0, F_n - 1, 1, 2, \underbrace{1, \ldots, 1}_{n-3}, 3, \underbrace{1, \ldots, 1}_{n-4}, 2] \text{ for } n > 3. \tag{5}$$

For any value of $n \geq 4$, (4) has $2n - 2$ terms and (5) has $2n - 1$ terms. Instead of (1), we could use [Dr. Göttsch, private communication]

$$\frac{1}{b} - \frac{1}{b^2 + b + a^2} = \left[0, b, 1, \frac{b^2}{a^2}\right] \text{ and } \frac{1}{b} + \frac{1}{2b^2 - b + a^2} = \left[0, b - 1, 1, 1, \frac{b^2}{a^2}\right].$$

This means one additional term 1 each. Letting $a = F_n, b = F_{n+1}$, we have $(b, b^2 + b + a^2) = 1$ and $(b, 2b^2 - b + a^2) = 1$. In the analogues of (4) and (5), we also have one additional term 1 each, namely,

$$\frac{1}{F_{n+1}} - \frac{1}{F_{n+1}^2 + F_{n+1} + F_n^2} = [0, F_{n+1}, 1, 2, \underbrace{1, \ldots, 1}_{n-3}, 3, \underbrace{1, \ldots, 1}_{n-4}, 2] \text{ for } n > 3$$

and

$$\frac{1}{F_{n+1}} + \frac{1}{2F_{n+1}^2 - F_{n+1} + F_n^2} = [0, F_{n+1} - 1, 1, 1, 2, \underbrace{1, \ldots, 1}_{n-3}, 3, \underbrace{1, \ldots, 1}_{n-4}, 2] \text{ for } n > 3.$$

This proves

Theorem: For every integer $m > 5$, resp. $m > 6$, there exist integers $b_m > a_m > 1$ with $(b_m, a_m) =$ 1, resp. $d_m > c_m > 1$ with $(d_m, c_m) = 1$, such that the simple continued fraction of $\frac{1}{a_m} - \frac{1}{b_m}$, resp. $\frac{1}{c_m} + \frac{1}{d_m}$, has exactly m terms.

By $\frac{1}{2} \mp \frac{1}{3}$, $\frac{1}{2} \mp \frac{1}{5}$, $\frac{1}{2} \mp \frac{1}{7}$, $\frac{1}{2} \mp \frac{1}{9}$, Theorem 1 holds for $m > 1$ and $m > 2$. We have

$$\phi_n = \frac{F_{n+2}}{F_{n+1}} = [\underbrace{1, \ldots, 1}_{n+1}] \text{ for } n \geq 0$$

without normalization. For every real $\tilde{\mu}$ between ϕ_{n-1} and ϕ_n, we have

$$\tilde{\mu} = [\underbrace{1, \ldots, 1}_{n}, \ldots] \text{ for } n > 0. \tag{6}$$

We also have

$$\phi-\phi_n=\frac{(-1)^n}{F_{n+1}(\phi F_{n+1}+F_n)},\quad |\phi-\phi_n|>\frac{1}{2\phi F_{n+1}^2}>\phi^{-2n-2}\ \text{ for } n>0.$$

Trivially, we observe that every real $\tilde{\mu}$ with

$$\phi<\tilde{\mu}<\phi+\phi^{-2n-2} \tag{7}$$

or with

$$\phi-\phi^{-2n-2}<\tilde{\mu}<\phi \tag{8}$$

satisfies (6).

For primes p, q, let $q>p^2+p$, $\mu=\frac{q-p}{p^2}$. Then we have $\mu>1$, $\frac{1}{p}-\frac{1}{q}=[0,p,\mu]$. $\tilde{\mu}=\mu$ should satisfy (7), which means $\phi p^2+p<q<\phi p^2+p+\phi^{-2n-2}p^2$. For $x>x_0$, there exist primes q with $x<q<x+x^{2/3}$, by Hoheisel (see [1]) and others. We use this with $x=\phi p^2+p$ and choose $p>x_0$ so that

$$\phi^{-2n-2}p^2\geq(\phi p^2+p)^{2/3};$$

by $\phi^2p^2>\phi p^2+p$, the choice $p>x_0+\phi^{3n+5}$ is sufficient. By the "Bertrand postulate" (and especially by Hoheisel), $p<2(x_0+\phi^{3n+5})$ can be satisfied. This proves

$$\underset{C\in\mathbb{R}_{>1}}{\exists}\ \underset{n>0}{\forall}\ \underset{\substack{p_n,q_n\in\mathbb{P}\\ p_n<q_n<C^n}}{\exists}\ \frac{1}{p_n}-\frac{1}{q_n}=[0,p_n,\underbrace{1,\ldots,1}_{n},\ldots]. \tag{9}$$

For primes p, q, let $q>p^2-p$, $\lambda=\frac{p+q}{p+q-p^2}$. Then we have $\lambda>1$, $\frac{1}{p}+\frac{1}{q}=[0,p-1,\lambda]$. $\tilde{\mu}=\lambda$ should satisfy (8), which means (after rewriting)

$$\phi^2p^2-p<q<\phi^2p^2-p+\phi^{-2n-1}(p+q-p^2). \tag{10}$$

Since $p+q-p^2>p+(\phi^2p^2-p)-p^2=\phi p^2$, the condition $\phi^2p^2-p<q<\phi^2p^2-p+\phi^{-2n}p^2$ is sufficient for (10). As above, we apply Hoheisel. This proves

$$\underset{C\in\mathbb{R}_{>1}}{\exists}\ \underset{n>0}{\forall}\ \underset{\substack{p_n,q_n\in\mathbb{P}\\ p_n<q_n<C^n}}{\exists}\ \frac{1}{p_n}+\frac{1}{q_n}=[0,p_n-1,\underbrace{1,\ldots,1}_{n},\ldots]. \tag{11}$$

In (9) and in (11), we have $q_n>F_n$.

On examining the argument, we see that p and q in (9) and also in (11) can be taken from arbitrary sets $\subset\mathbb{N}$ which satisfy conditions of types Bertrand and Hoheisel, respectively.

REFERENCE

1. K. Chandrasekharan. *Arithmetical Functions.* Berlin-Heidelberg-New York: Springer-Verlag 1970.

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