REDUCED ϕ-PARTITIONS OF POSITIVE INTEGERS*

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(Submitted January 1992)

1. INTRODUCTION

As a generalization of the equation $\phi(x)+\phi(k)=\phi(x+k)$, ϕ-partitions and reduced ϕ-partitions and reduced ϕ-partitions of positive integers were considered by Patricia Jones [1]. That is, $n=a_1+\cdots+a_i$ is a ϕ-partition if $i>1$ and $\phi(n)=\phi(a_1)+\cdots+\phi(a_i)$, where ϕ is Euler's totient function. Furthermore, a ϕ-partition is reduced if each of its summands is simple, where a simple number is known as 1 or a product of the first primes.

In [1] the author conjectured that every nonsimple number has exactly one reduced ϕ-partition. Here, we show that the conjecture is false. In fact, we will see that the positive integers satisfying the conjecture are quite rare. The main purpose of this paper is to give a complete characterization of positive integers that have exactly one reduced ϕ-partition.

Throughout the paper, let p and q denote distinct primes, especially, p_i denote the i^{th} prime, and $A_0=1$, $A_i=\prod_{p\le p_i} p$ be the i^{th} simple number.

It is shown in [1] that every simple number has no ϕ-partitions and every nonsimple number has a ϕ-partition as follows:

(I) $\quad n=\underbrace{p^{\alpha-1}t+\cdots+p^{\alpha-1}t}_{p}$ if $n=p^\alpha t$ for $\alpha>1$ and $p\nmid t$;

(II) $\quad n=\underbrace{j+\cdots+j}_{p-q}+qj$ if $n=pj$ where p and q do not divide j and $q<p$.

This gives algorithms from which we can obtain at least one reduced ϕ-partition of any non-simple number.

A nonsimple number is called semisimple if it has exactly one reduced ϕ-partition.

Our main result is the following:

Theorem: Let n be nonsimple. Then n is semisimple if and only if

(i) $\quad n$ is a prime or $n=3^2$, or

(ii) $\quad n=aq_1\cdots q_kA_i$ with $a(q_1-p_{i+1})\cdots(q_k-p_{i+1})<p_{i+1}$, where $i\gtrless 1$, $k\ge 0$, $q_1>q_2>\cdots>q_k>p_{i+1}$ are primes and a is a positive integer.

We will present the proof of the Theorem in Section 3.

It can be seen from the Theorem that $(p_{i+1}-1)A_i$ and $p_{i+2}A_i$ are semisimple. For $k\ge 2$, the smallest semisimple number is $2\times 3\times 5\times 7\times 11\times 13\times 19\times 23=19\times 23\times A_6$.

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* Research supported by the National Natural Science Foundation of China.

2. LEMMAS

First, we state without proof a basic and simple lemma.

Lemma 1: Let n be semisimple and $n = a_1 + \cdots + a_i$ be any of its ϕ-partitions. Then every a_i is simple or semisimple.

Lemma 2: Let n be odd. Then n is not semisimple except $n = p$ or 3^2.

Proof: Using the algorithms (I, II), we know that one of pq and p^α ($\alpha > 1$ and $p^\alpha > 3^2$) equals n, or a summand of some ϕ-partition of n. We have the reduced ϕ-partitions of pq and p^α as follows:

$$pq = \underbrace{1 + \cdots + 1}_{(p-2)(q-2)-2} + \underbrace{2 + \cdots + 2}_{p+q-1} = \underbrace{1 + \cdots + 1}_{(p-2)(q-2)} + \underbrace{2 + \cdots + 2}_{p+q-5} + 6,$$

$$p^\alpha = \underbrace{1 + \cdots + 1}_{p^{\alpha-1}(p-2)} + \underbrace{2 + \cdots 2}_{p^{\alpha-1}} = \underbrace{1 + \cdots + 1}_{p^{\alpha-1}(p-2)+2} + \underbrace{2 + \cdots + 2}_{p^{\alpha-1}-4} + 6.$$

Now the result follows from Lemma 1. □

Lemma 3: Suppose

$$n = \underbrace{1 + \cdots + 1}_{x_0} + \underbrace{A_1 + \cdots + A_1}_{x_1} + \cdots \underbrace{A_i + \cdots + A_i}_{x_i}$$

is a ϕ-partition. Then n is not semisimple if $x_j \geq p_{j+1} + 1$ for some $1 \leq j \leq i$.

Proof: It is sufficient to show that

$$(p_{j+1} + 1)A_j = \underbrace{A_j + \cdots + A_j}_{p_j+1}$$

is not the only reduced ϕ-partition of $(p_{j+1} + 1)A_j$.

Since $A_j / 2$ is not simple, it has a reduced ϕ-partition

$$A_j / 2 = \underbrace{1 + \cdots + 1}_{y_0} + \underbrace{A_1 + \cdots + A_1}_{y_1} + \cdots + \underbrace{A_{j-1} + \cdots + A_{j-1}}_{y_{j-1}}$$

which is obtained by algorithm (II). (Notice that $y_\ell \neq 0$ for $0 \leq \ell \leq j - 1$). Hence,

$$\phi(A_j) = \phi(A_j / 2) = y_0 + y_1\phi(A_1) + \cdots + y_{j-1}\phi(A_{j-1}).$$

It follows that

$$(p_{j+1} + 1)A_j = \underbrace{1 + \cdots + 1}_{2y_0} + \underbrace{A_1 + \cdots + A_1}_{2y_i} + \cdots + \underbrace{A_{j-1} + \cdots + A_{j-1}}_{2y_{j-1}} + A_{j+1} \tag{1}$$

is a reduced ϕ-partition. □

Lemma 4: Let $n = mA_i$ with $i > 1$, $p_{i+1} \nmid m$ and $p_{i+j}^2 | m$ for some $j > 1$. Then n is not semisimple.

Proof: Put $m' = m / p_{i+j}$. Then

$$n = \underbrace{m'A_i + \cdots + m'A_i}_{p_{i-j}}$$

is a ϕ-partition. Hence, if the reduced ϕ-partition

$$n = \underbrace{A_i + \cdots + A_i}_{x_i} + \underbrace{A_{i+1} + \cdots + A_{i+1}}_{x_{i+1}} + \cdots + \underbrace{A_{i+t} + \cdots + A_{i+t}}_{x_{i+t}}$$

is obtained by following the algorithms (I, II), then $x_i \ge p_{i+j} > p_{i+1}$. Thus, by Lemma 3, n is not semisimple. □

3. PROOF OF THE THEOREM

It is evident that primes and 3^2 are all semisimple. By Lemma 2 and Lemma 4, we need to consider only $n = aq_1 \cdots q_k A_i$ as given in the Theorem.

Write $q_j - p_{i+1} = \alpha_j$ for $i \le j \le k$ and $p_{i+2} - p_{i+1} = \beta$. Then $\alpha_1 > \alpha_2 > \cdots . \alpha_k$ and $\alpha_j > \beta$ for $1 \le j \le k-1$.

It is easy to see from the definition that n has a reduced ϕ-partition if and only if there are nonnegative integers $x_0, x_1, \ldots, x_\ell$ such that

$$\begin{cases} \quad n = x_0 + x_1 A_1 + \cdots + x_\ell A_\ell, \\ \phi(n) = x_0 + x_1 \phi(A_1) + \cdots + x_\ell \phi(A_\ell). \end{cases} \tag{2}$$

Further, n is semisimple if $(x_0, x_1, \ldots, x_\ell)$ is unique.

For $n = aq_1 \cdots q_k A_i$, we have a reduced ϕ-partition

$$\begin{cases} \quad n = a_i A_i + \cdots + a_{i+k} A_{i+k}, \\ \phi(n) = a_i \phi(A_1) + \cdots + a_{i+k} \phi(A_{i+k}), \end{cases} \tag{3}$$

which is obtained by the algorithm (II). On the other hand, we have the ϕ-partition

$$n = aq_1 \cdots q_k A_i = \underbrace{aq_1 \cdots q_{k-1} A_i + \cdots + aq_1 \cdots q_{k-1} A_i}_{\alpha_k} + aq_1 \cdots q_{k-1} A_{i+1}.$$

Let the reduced ϕ-partitions

$$\begin{cases} \quad q_1 \cdots q_{k-1} A_i = b_i A_i + \cdots + b_{i+k-1} A_{i+k-1}, \\ \phi(q_1 \cdots q_{k-1} A_i) = b_i \phi(A_i) + \cdots + b_{i+k-1} \phi(A_{i+k-1}), \end{cases} \tag{4}$$

and

$$\begin{cases} \quad q_1 \cdots q_{k-1} A_{i+1} = c_{i+1} A_{i+1} + \cdots + c_{i+k} A_{i+k}, \\ \phi(q_1 \cdots q_{k-1} A_{i+1}) = c_{i+1} \phi(A_{i+1}) + \cdots + c_{i+k} \phi(A_{i+k}), \end{cases} \tag{5}$$

be obtained by the algorithm (II). Then $a_i = ab_i \alpha_k, a_{i+j} = a(b_{i+j} \alpha_k + c_{i+j})$ for $1 \le j \le k-1$ and $a_{i+k} = ac_{i+k}$. It is not difficult to show by induction on k that

$$a_i = a\alpha_1 \cdots \alpha_k, b_i = \alpha_1 \cdots \alpha_{k-1} \text{ and } c_{i+1} = (\alpha_1 - \beta) \cdots (\alpha_{k-1} - \beta).$$

We now proceed by induction on k to prove that $a_i > a_{i+1} > \cdots > a_{i+k}$. When $k = 0$, there is nothing to show. Suppose that $k > 0$ and the conclusion holds for $k - 1$. From this, we can assume that

$$b_i > b_{i+1} > \cdots > b_{i+k-1} \text{ and } c_{i+1} > \cdots > c_{i+k}.$$

Thus,

$$a_{i+j} - a_{i+j+1} = a[(b_{i+j} - b_{i+j-1})\alpha_k + c_{i+j} - c_{i+j+1}] > 0 \text{ for } 1 \le j \le k-1.$$

It remains to show that $a_i > a_{i+1}$. We claim that $a_i = \beta a_{i+1} + a(\alpha_1 - \beta)\cdots(\alpha_k - \beta)$ which implies the conclusion. In fact, it is obvious for $k = 1$. Assume it holds for $k - 1 > 0$. From this, it follows that $b_i = \beta b_{i+1} + (\alpha_1 - \beta)\cdots(\alpha_{k-1} - \beta) = \beta b_{i+1} + c_{i+1}$. Thus, $a_i = ab_i\alpha_k = a(\beta b_{i+1} + c_{i+1})\alpha_k$ $= a(\beta b_{i+1}\alpha_k + \beta c_{i+1}) + ac_{i+1}(\alpha_k - \beta)$ $= \beta a_{i+1} + a(\alpha_1 - \beta)\cdots(\alpha_k - \beta)$. Recall that $a_i < p_{i+1}$.

Set

$$S = S(n) = \{\underline{x} = (x_0, x_1, \ldots, x_{i+k}) \mid \underline{x} \text{ satisfies } (2)\}.$$

Then $\underline{a} = (a_0, \ldots, a_{i-1}, a_i, \ldots, a_{i+k}) \in S$, where $a_0 = \cdots a_{i-1} = 0$ and $a_i, \ldots, a_{i+k}$ are as in (3). Define on S an order "$\succ$" as $\underline{x} \succ \underline{x'}$ if $x_j > x'_j$, for some $j \ge 0$, and $x_{j+\ell} \ge x'_{j+\ell}$ for $\ell \ge 0$. Since

$$\begin{aligned} n = \sum_{j=i}^{i+k} a_j A_j &\le \sum_{j=i}^{i+k-1} (p_{j+1} - 1)A_j + a_{i+k}A_{i+k} \\ &= -A_i + (a_{i+k} + 1)A_{i+k} < (a_{i+k} + 1)A_{i+k} < A_{i+k+1}, \end{aligned}$$

every solution of (2) is contained in S, and similarly, we can show that $\underline{a}$ is the maximal element of the totally ordered set $(S, \succ)$. If $S \ne \{\underline{a}\}$, we let $\underline{b}$ be the maximal element of $(S \setminus \{\underline{a}\}, \succ)$ and distinquish two cases as follows:

(i) $b_j > p_{j+1}$ for some $1 \le j \le i + k$. Put

$$\underline{t} = (b_0 + y_0, b_1 + y_1, \ldots, b_{j-1} + y_{j-1}, b_j, b_{j+1} + 1, \ldots, b_{i+k})$$

where $y_0, y_1, \ldots, y_{j-1}$ are as in (1). Then it follows that $\underline{t} \in S$. Since $\underline{t} > \underline{b}$, then $\underline{t} = \underline{a}$. In fact, this is impossible since, in formula (1), $y_\ell \ne 0, \ell = 0, 1, \ldots, j-1$, always holds. This contradicts $a_0 = 0$.

(ii) $b_j \le p_{j+1}, j = 1, 3, \ldots, i+k$. Since $\underline{a} \succ \underline{b}$, there is an $\ell, i \le \ell \le i+k$, such that $a_\ell > x_\ell$ and $a_{\ell+j} = b_{\ell+j}$ for $j > 0$. Write $c = a_\ell - x_\ell^0$ and $c_j = x_j^0 - a_j, j = 0, 1, \ldots, \ell - 1$. Then

$$cA_\ell = \sum_{j=0}^{\ell-1} c_j A_j \text{ and } c\phi(A_\ell) = \sum_{j=0}^{\ell-1} c_j \phi(A_j).$$

Thus,

$$c\big(A_\ell - \phi(A_t)\big) = \sum_{j=1}^{\ell-1} c_j\big(A_j - \phi(A_j)\big).$$

Set $\sigma_j = \phi(A_j)/A_j$. Then $\sigma_j > \sigma_{j+1}$ for $j \ge 1$, and $0 < (1-\sigma_j)/(1-\sigma_\ell) < 1$ for $1 \le j < \ell$. Put $\tau_j = (1-\sigma_j)/(1-\sigma_\ell)$. Then

$$cA_\ell = \sum_{j=1}^{\ell-1} c_j A_j \tau_j \le \sum_{j=1}^{\ell-1} |c_j| A_j \tau_j < \sum_{j=1}^{\ell-1} |c_j| A_j.$$

If $\ell = i$ (when $k = 0$ this is always the case), then $c_j = x_j^0$ for $0 \le j < \ell$. In this case,

$$cA_\ell < \sum_{j=1}^{\ell-1} |c_j| A_j = \sum_{j=1}^{\ell-1} c_j A_j \le cA_\ell,$$

which is a contradiction. If $\ell > i$, then $a_{\ell-1} > a_\ell \ge 1$, and

$$cA_\ell < \sum_{j=1}^{\ell-1} |c_j| A_j \le (p_\ell - 2)A_{\ell-1} + \sum_{j=1}^{\ell-2} p_{j+1} A_j = A_\ell - A_{\ell-1} + A_{\ell-2} + \cdots + A_2 < A_\ell$$

which again yields a contradiction. By the preceding discussion, we have shown $S = \{\underline{a}\}$, i.e., $\underline{a}$ is unique. The proof is complete. □

4. CONCLUDING REMARKS

We mention here that it would be interesting to find the set $S(n)$ for any nonsemisimple number n. We guess that there is a unique $\underline{x} = (x_0, x_1, \ldots)$ in $S(n)$ such that $o \le x_j \le p_{j+1}$ for $j \ge 1$. In this case, $S(n)$ can be derived exclusively by using the algorithms (I, II) and formula (1).

ACKNOWLEDGMENT

I wish to thank the referee for a shorter and simpler proof of Lemma 3 and for many valuable comments.

REFERENCE

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