ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-488 ***Proposed by Paul S. Bruckman, Highwood, IL***

The *Fibonacci pseudoprimes* (or FPP's) are those composite integers n with $\gcd(n, 10) = 1$ and satisfying the following congruence:

$$F_{n-\varepsilon_n} \equiv 0 \pmod{n}, \tag{i}$$

where

$$\varepsilon_n = \begin{cases} 1 & \text{if } n \equiv \pm 1 \pmod{10}, \\ -1 & \text{if } n \equiv \pm 3 \pmod{10}. \end{cases}$$

$$\left[\text{Thus, } \varepsilon_n = \left(\tfrac{5}{n}\right), \text{ a Jacobi symbol.}\right]$$

Given a prime $p > 5$, prove that $u = \frac{1}{3}L_{2p}$ is a FPP if u is composite.

The *Lucas pseudoprimes* (or LPP's) are those composite positive integers n satisfying the following congruence:

$$L_n \equiv 1 \pmod{n}. \tag{ii}$$

Given a prime $p > 5$, prove that $u = \frac{1}{3}L_{2p}$ is a LPP if u is composite.

H-489 ***Proposed by H.-J. Seiffert, Berlin, Germany***

Define the sequences of Pell numbers and Pell-Lucas numbers by

$$P_0 = 0,\ P_1 = 1,\ P_{k+2} = 2P_{k+1} + P_k,$$
$$Q_0 = 2,\ Q_1 = 2,\ Q_{k+2} = 2Q_{k+1} + Q_k.$$

Show that

(a) $$\sum_{n=1}^{\infty} \frac{F_{2^n}Q_{2^n}}{8(L_{2^n}P_{2^n})^2 - 5(F_{2^n}Q_{2^n})^2} = \frac{1}{12},$$

(b) $$\sum_{n=1}^{\infty} \frac{L_{2^n}P_{2^n}}{8(L_{2^n}P_{2^n})^2 - 5(F_{2^n}Q_{2^n})^2} = \frac{8-3\sqrt{2}}{48}.$$

SOLUTIONS

A Soft Matrix

H-474 ***Proposed by R. Andre-Jeannin, Longwy, France***
(Vol. 31, no. 1, February 1993)

Let us define the sequence $\{U_n\}$ by

$$U_0 = 0,\ U_1 = 1, \text{ and } U_n = PU_{n-1} - QU_{n-2},\ n \in \mathbb{Z},$$

where P and Q are nonzero integers. Assuming that $U_k \neq 0$, the matrix M_k is defined by

$$M_k = \frac{1}{U_k}\begin{pmatrix} U_{k+1} & iQ^{k/2} \\ iQ^{k/2} & -Q^k U_{1-k} \end{pmatrix},\ k \geq 1,$$

where $i = \sqrt{(-1)}$.

Express in a closed form the matrix M_k^n, for $n \geq 0$.

Reference: A. F. Horadam & P. Filipponi, "Choleski Algorithm Matrices of Fibonacci Type and Properties of Generalized Sequences," *The Fibonacci Quarterly* **29.2** (1991):164-73.

Solution by H.-J. Seiffert, Berlin, Germany

First, we prove that for all integers m, h, and j,

$$U_{m+h}U_{m+j} - U_m U_{m+h+j} = Q^m U_h U_j. \tag{1}$$

We consider the Fibonacci polynomials defined by

$$F_0(x) = 0,\ F_1(x) = 1,\ F_n(x) = xF_{n-1}(x) + F_{n-2}(x),\ n \in \mathbb{Z}.$$

It is easily seen that

$$U_n = (-Q)^{(n-1)/2} F_n(x),\ n \in \mathbb{Z}, \tag{2}$$

where $x = P/\sqrt{-Q}$. Multiplying the well-known equation [see A. F. Horadam & Bro. J. M. Mahon, "Pell and Pell-Lucas Polynomials," *The Fibonacci Quarterly* **23.1** (1985):12, formula (3.32), where the polynomials $P_k(x) = F_k(2x)$ are considered]

$$F_{m+h}(x)F_{m+j}(x) - F_m(x)F_{m+h+j}(x) = (-1)^m F_h(x)F_j(x)$$

by $(-Q)^{m-1+(h+j)/2}$ and regarding (2), we obtain (1). From (2), it also follows that

$$U_{-n} = -Q^{-n}U_n,\ n \in \mathbb{Z}. \tag{3}$$

For $m = k$, $h = 1$, and $j = n$, (1) yields

$$U_{k+1}U_{k+n} - Q^k U_n = U_k U_{k+n+1}. \tag{4}$$

Similarly, with $m = k$, $h = n - k$, and $j = 1$,

$$U_n U_{k+1} - Q^k U_{n-k} = U_k U_{n+1}, \tag{5}$$

with $m = k$, $h = n$, and $j = 1 - k$,

$$U_{k+n} - Q^k U_n U_{1-k} = U_k U_{n+1}, \tag{6}$$

and finally, with $m = n$, $h = 1-k$, and $j = k-n$, (1) gives

$$U_n + Q^n U_{1-k} U_{k-n} = U_k U_{n+1-k}$$

or, by (3),

$$U_n - Q^k U_{1-k} U_{n-k} = U_k U_{n+1-k}. \quad (7)$$

With the help of (4)-(7), it is easily proved by induction on n that

$$M_k^n = \frac{1}{U_k}\begin{pmatrix} U_{k+n} & iQ^{k/2}U_n \\ iQ^{k/2}U_n & -Q^k U_{n-k} \end{pmatrix}, \quad n \geq 1.$$

Using (3), it is easily seen that this equation also holds for $n = 0$.

Also solved by P. Bruckman, A. G. Shannon, and the proposer.

Get It off Your Chess

H-475 *Proposed by Larry Taylor, Rego Park, NY*
(Vol. 31, no. 2, May 1993)

Professional chess players today use the algebraic chess notation. This is based upon the algebraic numbering of the chessboard. The eight letters *a* through *h* and the eight digits 1 through 8 are used to form sixty-four combinations of a letter and a digit which are called "symbol pairs." Those sixty-four symbol pairs are used to represent the sixty-four squares of the chessboard.

Develop a viable arithmetic numbering of the chessboard, as follows:

(a) Use twenty-five letters of the alphabet (all except *U*) and nine decimal digits (all except zero) to form 225 symbol pairs; choose sixty-four of those symbol pairs to represent the sixty-four squares of the chessboard.

(b) There are thirty-six squares from which a King can move to eight other squares. Let the nine symbol pairs representing the location of the King and the squares to which it can move contain all nine decimal digits.

(c) There are sixteen squares from which a Knight can move to eight other squares. A Queen located on one of those sixteen squares, moving one or two squares, can go to sixteen other squares. Let the twenty-five symbol pairs representing the location of the Knight or the Queen and the squares to which the Knight or the Queen can move contain all twenty-five letters of the alphabet.

(d) Let the algebraic Square *a*8 (the original location of Black's Queen Rook) correspond to the arithmetic Square *A*1; let the algebraic Square *h*1 (the original location of White's King Rook) correspond to the arithmetic Square *Z*9.

Solution by Leonard A. G. Dresel, Reading, England

Consider the basic 3×3 and 5×5 patterns given by:

				A	B	C	D	E
1	2	3		F	G	H	I	J
4	5*	6	and	K	L	M*	N	O
7	8	9*		P	Q	R	S	T
				V	W	X	Y	Z*

Repeating these patterns across the 8×8 board, left to right and then top to bottom, and superposing them, we can satisfy conditions (b) and (c). To satisfy (d) and obtain $Z9$ in the bottom right corner, we exchange 5 with 9 and M with Z in the basic patterns. Thus, we arrive at a viable numbering given by:

A1	B2	C3	D1	E2	A3	B1	C2
F4	G9	H6	I4	J9	F6	G4	H9
K7	L8	Z5	N7	O8	K5	L7	Z8
P1	Q2	R3	S1	T2	P3	Q1	R2
V4	W9	X6	Y4	M9	V6	W4	X9
A7	B8	C5	D7	E8	A5	B7	C8
F1	G2	H3	I1	J2	F3	G1	H2
K4	L9	Z6	N4	O9	K6	L4	Z9

Since 3 and 5 are co-prime, the repeating patterns ensure that no alpha-numeric combination occurs more than once.

The solution is not unique, as we can choose the modified basic patterns in $(7!) \times (23!)$ ways to satisfy condition (d).

Also solved by P. Bruckman, J. Hendel, and the proposer.

Pell-Mell

H-476 ***Proposed by H.-J. Seiffert, Berlin, Germany (Vol. 31, no. 2, May 1993)***

Define the Pell numbers by $P_0 = 0, P_1 = 1, P_n = 2P_{n-1} + P_{n-2}$, for $n \geq 2$. Show that, for all positive integers n,

$$P_n = \sum_{\substack{k=0 \\ 4 \nmid 2n+k}}^{n-1} (-1)^{[(3k+3-2n)/4]} 2^{[3k/2]} \binom{n+k}{2k+1},$$

where [ ] denotes the greatest integer function.

Solution by Paul S. Bruckman, Highwood, IL

Let S_n denote the sum given in the statement of the problem. It is easily shown that

$$\frac{x}{f(x)} = \sum_{n=1}^{\infty} P_n x^n, \; |x| < \sqrt{2} - 1, \tag{1}$$

where

$$f(x) = 1 - 2x - x^2. \tag{2}$$

To prove that $S_n = P_n$, $n = 1, 2, \ldots$, it will suffice to show that $g(x) = \frac{x}{f(x)}$, where

$$g(x) = \sum_{n=1}^{\infty} S_n x^n; \tag{3}$$

presumably, this is to be valid for all x with $|x| < \sqrt{2} - 1$.

As usual with generating function techniques, we will ignore questions of convergence (which *should* be considered, *a posteriori*). Then

$$g(x) = \sum_{\substack{k,\, m \geq 0 \\ 4 \nmid 2m+3k+2}} x^{m+k+1} (-1)^{\left[\frac{1}{4}(k+1-2m)\right]} 2^{[3k/2]} \binom{m+2k+1}{2k+1}.$$

Letting $m = 2u$ or $m = 2u+1$, we obtain

$$\begin{aligned} g(x) &= \sum_{\substack{k,u\geq 0\\ 4\nmid 3k+2}} x^{2u+k+1}(-1)^{[(k+1)/4]+u}2^{[3k/2]}\binom{2u+2k+1}{2u} \\ &\quad + \sum_{\substack{k,u\geq 0\\ 4\nmid 3k}} x^{2u+k+2}(-1)^{[(k-1)/4]+u}2^{[3k/2]}\binom{2u+1+2k+1}{2u+1} \\ &= \sum_{\substack{j\geq 1\\ 4\nmid j+1}} x^j(-1)^{[j/4]}2^{\left[\frac{3}{2}(j-1)\right]}\sum_{u\geq 0}(-1)^u x^{2u}\binom{-2j}{2u} \\ &\quad - \sum_{\substack{j\geq 1\\ 4\nmid j-1}} x^j(-1)^{\left[\frac{1}{4}(j-2)\right]}2^{\left[\frac{3}{2}(j-1)\right]}\sum_{u\geq 0}(-1)^u x^{2u+1}\binom{-2j}{2u+1}. \end{aligned}$$

Now

$$\sum_{u\geq 0}(-1)^u x^{2u}\binom{-2j}{2u} = \sum_{v\geq 0}(ix)^v e_v\binom{-2j}{v},$$

where $e_v = \frac{1}{2}(1+(-1)^v)$, which equals $\frac{1}{2}(\theta^{-2j}+\overline{\theta}^{-2j}) = \operatorname{Re}(\theta^{-2j})$, with $\theta \equiv 1+ix$. Likewise,

$$\sum_{u\geq 0}(-1)^u x^{2u+1}\binom{-2j}{2u+1} = -i\sum_{v\geq 0}(ix)^v o_v\binom{-2j}{v},$$

where $o_v = \frac{1}{2}(1-(-1)^v)$, which equals $\frac{1}{2i}(\theta^{-2j}-\overline{\theta}^{-2j}) = \operatorname{Im}(\theta^{-2j})$. Therefore,

$$g(x) = \operatorname{Re}(U(x)+iV(x)), \tag{4}$$

where

$$U(x) = \sum_{\substack{j\geq 1\\ 4\nmid j+1}} x^j(-1)^{\left[\frac{1}{4}j\right]}2^{\left[\frac{3}{2}(j-1)\right]}\theta^{-2j}, \tag{5}$$

$$V(x) = \sum_{\substack{j\geq 1\\ 4\nmid j-1}} x^j(-1)^{\left[\frac{1}{4}(j-2)\right]}2^{\left[\frac{3}{2}(j-1)\right]}\theta^{-2j}. \tag{6}$$

Making the substitutions $j = 4i+r$, where $i \geq 0$ and $r = 1, 2,$ or 4 in (5), $r = 2, 3, 4$ in (6), we find that

$$U(x) = (x/\theta^2 + 2x^2/\theta^4 - 16x^4/\theta^8)\cdot h(x), \tag{7}$$

$$V(x) = (2x^2/\theta^4 + 8x^3/\theta^6 + 16x^4/\theta^8)\cdot h(x), \tag{8}$$

where

$$h(x) = \sum_{i=0}^{\infty}(-1)^i 2^{6i}x^{4i}\theta^{-8i}. \tag{9}$$

Thus,

$$h(x) = (1+64x^4/\theta^8)^{-1} = \frac{\theta^8}{\theta^8+64x^4},$$

from which we obtain

$$U(x) = \frac{x}{\theta^8+64x^4}\cdot(\theta^6+2x\theta^4-16x^3), \tag{10}$$

$$V(x) = \frac{2x^2}{\theta^8 + 64x^4} \cdot (\theta^4 + 4x\theta^2 + 8x^2). \tag{11}$$

As we may verify, $\theta^8 + 64x^4 = (\theta^4 + 4x\theta^2 + 8x^2)(\theta^4 - 4x\theta^2 + 8x^2)$ and $\theta^6 + 2x\theta^4 - 16x^3 = (\theta^4 + 4x\theta^2 + 8x^2)(\theta^2 - 2x)$. Thus,

$$U(x) = \frac{x(\theta^2 - 2x)}{\theta^4 - 4x\theta^2 + 8x^2}, \quad V(x) = \frac{2x^2}{\theta^4 - 4x\theta^2 + 8x^2}. \tag{12}$$

Next, we observe that $\theta^2 = 1 + 2ix - x^2 = 1 - 2x - x^2 + 2x(1+i) = f(x) + 2x(1+i)$. Also, we have $\theta^4 = (f(x))^2 + 4xf(x) \cdot (1+i) + 4x^2 \cdot 2i = f(x)[f(x) + 4ix] + 4xf(x) + 8ix^2$, from which it follows that $\theta^4 - 4x\theta^2 + 8x^2 = f(x)[f(x) + 4ix]$. Then

$$U(x) + iV(x) = \frac{x(\theta^2 - 2x + 2ix)}{f(x)[f(x) + 4ix]} = \frac{x[f(x) + 4ix]}{f(x)[f(x) + 4ix]} = \frac{x}{f(x)}.$$

Hence, we see that $U(x) + iV(x)$ is real, so that

$$\text{Re}(U(x) + iV(x)) = U(x) + iV(x) = g(x) = \frac{x}{f(x)}. \quad \text{Q.E.D.}$$

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