CONGRUENCES AND RECURRENCES FOR BERNOULLI NUMBERS OF HIGHER ORDER

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1. INTRODUCTION

The Bernoulli polynomials of order k, for any integer k, may be defined by (see [10], p. 145):

$$\frac{x^k e^{xz}}{(e^x-1)^k}=\sum_{n=0}^{\infty} B_n^{(k)}(z)\frac{x^n}{n!}. \tag{1.1}$$

In particular, $B_n^{(k)}(0)=B_n^{(k)}$, the Bernoulli number of order k, and $B_n^{(1)}=B_n$, the ordinary Bernoulli number. Note also that $B_n^{(0)}=0$ for $n>0$.

The polynomials $B_n^{(k)}(z)$ and the numbers $B_n^{(k)}$ were first defined and studied by Niels Nörlund in the 1920s; later they were the subject of many papers by L. Carlitz and others. For the past twenty-five years not much has been done with them, although recently the writer found an application for $B_n^{(k)}$ involving congruences for Stirling numbers (see [8]). For the writer, the higher-order Bernoulli polynomials and numbers are still of interest, and they are worthy of further investigation.

Apparently, not much is known about the divisibility properties of $B_n^{(k)}$ for general k. Carlitz [2] proved that if p is prime and

$$k=a_1p^{k_1}+a_2p^{k_2}+\cdots+a_rp^{k_r}\ \ (0\le k_1<k_2<\cdots k_r;\ 0<a_j<p),$$

then $p^rB_n^{(k)}$ is integral (mod p) for all n. He (see [4], [5]) also proved the following congruences for primes $p>3$:

$$B_p^{(p)}\equiv-\frac{1}{2}p^2(p-1)!\ (\text{mod}\ p^5), \tag{1.2}$$

$$B_{p+2}^{(p+1)}\equiv\frac{1}{6}p^3\ \ (\text{mod}\ p^4), \tag{1.3}$$

$$B_{p+2}^{(p)}\equiv\frac{1}{p+1}p^2B_{p+1}\ \ (\text{mod}\ p^4), \tag{1.4}$$

where B_{p+1} is the ordinary Bernoulli number. F. R. Olson [11] was able to extend (1.2) and (1.3) slightly by proving congruences modulo p^6 and p^5, respectively. Carlitz [4] proved that $B_n^{(p)}$ is integral (mod p), $p\ge3$, unless $n\equiv0\ (\text{mod}\ p-1)$ and $n\equiv0$ or $p-1\ (\text{mod}\ p)$, in which case $pB_n^{(p)}$ is integral. He also proved congruences for special cases of $B_n^{(p)}$.

The writer [8] examined the numbers $B_n^{(n)}$ and proved that, for p prime, $p>3$, r odd, and $p+1\ge r\ge5$,

$$B_p^{(p)}=-\sum_{j=1}^{r-4}\frac{1}{j+1}s(p,j)p^{j+1}\ \ (\text{mod}\ p^r), \tag{1.5}$$

where $s(p, j)$ is the Stirling number of the first kind. (The Stirling numbers are defined in section 2.) This enables us to extend (1.2), theoretically, to any modulus p^r. Many other properties of $B_n^{(n)}$ are worked out in [8], and applications are given that involve new congruences for the Stirling numbers.

The purpose of the present paper is to examine the divisibility properties of $B_n^{(k)}$ for arbitrary n and k. We are able to extend congruences (1.3) and (1.4), and we also generalize many of the results in [8] and [10]. A summary of the main results follows.

1. We prove that the Bernoulli polynomials have the following property:

$$B_{n+k}^{(n)}\left(z+\frac{1}{2}n\right)=(-1)^{n+k}B_{n+k}^{(n)}\left(-z+\frac{1}{2}n\right).$$

To the writer's knowledge, this is a new result. It is very helpful in proving congruences (1.6)-(1.9) below.

2. We extend (1.3) and (1.4) by proving, for $p>5$:

$$B_{p+2}^{(p+1)}\equiv-\frac{1}{12}(p+2)!\,p^2 \pmod{p^6}, \tag{1.6}$$

$$B_{p+2}^{(p)}\equiv\frac{1}{24}p^2(p+2)!(p+12b_{p+1}) \pmod{p^7}, \tag{1.7}$$

$$B_{p+4}^{(p)}\equiv\frac{1}{12}p^2(p+4)!(3p+2)b_{p+3} \pmod{p^4}, \tag{1.8}$$

where b_n is the Bernoulli number of the second kind, defined and studied by Jordan [9], pp. 265-287 and by Carlitz [1]. The numbers b_n are also defined in section 2 of this paper, and we show in section 2 that $B_n^{(n-1)}=-(n-1)n!\,b_n$

3. Motivated by (1.6), we prove that if n is odd and composite, $n>9$, then

$$B_{n+2}^{(n+1)}\equiv 0 \pmod{n^4}. \tag{1.9}$$

4. For $k\geq 0$, we define

$$A_k(p;n)=\frac{(-1)^n p^{[n/(p-1)]}}{n!}B_n^{(n-k)},$$

and we prove that $A_k(p;n)$ is integral (mod p); in fact, if p does not divide k, then $\frac{1}{n-k}A_k(p;n)$ is integral (mod p). This improves results of Carlitz [2], [3].

5. With $A_k(p;n)$ as defined above, we prove

$$A_k(p;r(p-1))\equiv(-1)^r\binom{r+k}{k} \pmod{p}, \tag{1.10}$$

$$A_k(p;r(p-1)+1)\equiv\frac{1}{2}(-1)^{r-1}(r+k-1)\binom{r+k}{k} \pmod{p}\quad (p>2). \tag{1.11}$$

These congruences give us some insight into the highest power of p (especially $p=2$) dividing the denominator of $B_n^{(n-k)}$. This is discussed in sections 3 and 4.

6. We prove the following recurrence formulas, which generalize results of Nörlund [10], p. 150, for $k=0$. For $k \geq 0$,

$$\frac{B_n^{(n-k+1)}}{n!} = \sum_{r=0}^{n} \frac{(-1)^{n-r}}{n+1-r} \frac{B_r^{(r-k)}}{r!},$$

$$\frac{(-1)^{n+k} B_{n+k}^{(n)}}{(n+k)!} = \sum_{r=0}^{n} \binom{n}{r} \frac{B_{r+k}^{(r)}}{(r+k)!}.$$

These recurrences turn out to be helpful in proving (1.10), (1.11), and the fact that $A_k(p;n)$ is integral (mod p).

Section 2 is a preliminary section that includes the definitions and known results that we need. In section 3 we examine $B_n^{(k)}$ for arbitrary n and k, and we find new congruences, generating functions and recurrences. In section 4 we look at $B_n^{(n-1)}$ in more detail, and we find some additional properties.

Throughout the paper, the letter p designates a prime number and the letter n denotes a nonnegative integer.

2. PRELIMINARIES

We first note some special cases (see [4]). If $n < k$, then $B_n^{(k)} = \binom{k-1}{n}^{-1} s(k, k-n)$, where $s(k, k-n)$ is the Stirling number of the first kind, defined by

$$x(x-1)\cdots(x-n+1) = \sum_{k=0}^{n} s(n,k) x^k, \tag{2.1}$$

or by the generating function

$$\{\log(1+x)\}^k = k! \sum_{n=k}^{\infty} s(n,k) \frac{x^n}{n!}.$$

If $k > 0$, then $B_n^{(-k)} = \binom{n+k}{k}^{-1} S(n+k, k)$, where $S(n+k, k)$ is the Stirling number of the second kind, defined by

$$x^n = \sum_{k=0}^{n} S(n,k) x(x-1)\cdots(x-k+1),$$

or by the generating function

$$(e^x - 1)^k = k! \sum_{n=k}^{\infty} S(n,k) \frac{x^n}{n!}.$$

Since the Stirling numbers are well known and have been extensively studied (see, e.g., [6], ch. 5; [8]; and [9], ch. 4), in this paper we will concentrate on $B_n^{(k)}$ for $0 \leq k \leq n$.

It follows from (1.1) that (see [10], p. 150):

$$B_n^{(k)}(x+y) = \sum_{j=0}^{n} \binom{n}{j} x(x-1)\cdots(x-j+1) B_{n-j}^{(k-j)}(y) = \sum_{j=0}^{n} \binom{n}{j} B_{n-j}^{(k)}(y) x^j. \tag{2.2}$$

$$\frac{d}{dz} B_n^{(k)}(z) = n B_{n-1}^{(k)}(z), \tag{2.3}$$

$$B_n^{(k)}(z+1) - B_n^{(k)}(z) = n B_{n-1}^{(k-1)}(z). \tag{2.4}$$

Nörlund [10], p. 145, proved

$$B_n^{(k+1)}(z) = \left(1 - \frac{n}{k}\right) B_n^{(k)}(z) + (z-k)\frac{n}{k} B_{n-1}^{(k)}(z),$$

so that

$$B_n^{(n-k)} = \frac{k-n}{k} B_n^{(n-k+1)} + \frac{(k-n)n}{k} B_{n-1}^{(n-k)}. \tag{2.5}$$

Nörlund [10], p. 148, also proved

$$B_n^{(k)} = k\binom{n}{k}\sum_{r=0}^{k-1}(-1)^{k-1-r} s(k, k-r)\frac{B_{n-r}}{n-r}, \tag{2.6}$$

which is the basis for some of the results of Carlitz [3], such as (1.2)-(1.4) and the congruences for $B_n^{(p)}$. In (2.6), B_{n-r} is the ordinary Bernoulli number.

Nörlund [10], p. 147, proved the following integration formulas

$$B_n^{(n)}(x) = \int_x^{x+1}(t-1)(t-2)\cdots(t-n)\,dt, \tag{2.7}$$

$$B_{n+1}^{(n)} = -n\int_0^1 t(t-1)\cdots(t-n)\,dt, \tag{2.8}$$

which, when compared with (2.1), indicate the close relationship between the Stirling numbers and the higher-order Bernoulli numbers. Nörlund [10], pp. 147 and 150, also gave the following generating functions:

$$\frac{x^k}{\{\log(1+x)\}^k} = -k\sum_{\substack{n=0\\(n\neq k)}}^{\infty}\frac{B_n^{(n-k)}}{n-k}\frac{x^n}{n!}, \tag{2.9}$$

$$\frac{x}{(1+x)\log(1+x)} = \sum_{n=0}^{\infty} B_n^{(n)}\frac{x^n}{n!}. \tag{2.10}$$

Jordan [9], pp. 265-87, defined and studied b_n, the Bernoulli number of the second kind. The generating function is

$$\frac{x}{\log(1+x)} = \sum_{n=0}^{\infty} b_n x^n. \tag{2.11}$$

Comparing (2.9), (2.10), and (2.11), we see that, for $n \neq 1$,

$$\frac{1}{1-n}B_n^{(n-1)} = n!\,b_n = B_n^{(n)} + nB_{n-1}^{(n-1)}. \tag{2.12}$$

The last equality also holds when $n = 1$. To the writer's knowledge, this relationship between b_n and $B_n^{(n-1)}$ has not been pointed out before.

Jordan [9], p. 265, defined the polynomial $\Psi_n(z)$, which has the generating function

$$\frac{x(1+x)^z}{\log(1+x)} = \sum_{n=0}^{\infty}\Psi_n(z)x^n, \tag{2.13}$$

and he proved

$$\Psi_n\left(z-1+\frac{1}{2}n\right)=(-1)^n\Psi_n\left(-z-1+\frac{1}{2}n\right). \tag{2.14}$$

Carlitz [1] extended (2.13) in the logical way by defining $\beta_n^{(k)}(z)$:

$$\frac{x^k(1+x)^z}{\{\log(1+x)\}^k}=\sum_{n=0}^{\infty}\beta_n^{(k)}(z)\frac{x^n}{n!}. \tag{2.15}$$

Thus, $\beta_n^{(k)}(z)$ is analogous to $B_n^{(k)}(z)$, and $\beta_n^{(1)}(z)=n!\Psi_n(z)$. Carlitz also proved the very useful result,

$$\beta_n^{(k+1)}(z-1)=B_n^{(n-k)}(z). \tag{2.16}$$

Note that by (2.9), (2.12), (2.15), and (2.16) we have

$$B_n^{(n-k)}(1)=\beta_n^{(k+1)}(0)=\frac{k+1}{k+1-n}B_n^{(n-k-1)};\ B_n^{(n)}(1)=n!b_n. \tag{2.17}$$

3. $B_n^{(k)}$ for $0\le k\le n$

We first prove a theorem that is the basis for many of our later results.

Theorem 3.1: For all nonnegative integers k,

$$B_{n+k}^{(n)}\left(z+\frac{1}{2}n\right)=(-1)^{n+k}B_{n+k}^{(n)}\left(-z+\frac{1}{2}n\right). \tag{3.1}$$

Proof: We use induction on k. The theorem is true for $k=0$, since by (2.14) and (2.16) we have

$$B_n^{(n)}\left(z+\frac{1}{2}n\right)=\beta_n^{(1)}\left(z-1+\frac{1}{2}n\right)=n!\Psi_n\left(z-1+\frac{1}{2}n\right)$$
$$=(-1)^n n!\Psi_n\left(-z-1+\frac{1}{2}n\right)=(-1)^n B_n^{(n)}\left(-z+\frac{1}{2}n\right).$$

Assume (3.1) holds for a fixed $k-1$, i.e.,

$$B_{n+k-1}^{(n)}\left(z+\frac{1}{2}n\right)=(-1)^{n+k-1}B_{n+k-1}^{(n)}\left(-z+\frac{1}{2}n\right).$$

Then, if $n+k$ is even, $B_{n+k-1}^{(n)}(z+\frac{1}{2}n)$ is an odd function of z. By (2.3), this implies $B_{n+k}^{(n)}(z+\frac{1}{2}n)$ is an even function of z. That is, (3.1) holds for $n+k$ even. If $n+k$ is odd, then $n+1+k$ is even, and we apply the operator Δ to both sides of

$$B_{n+1+k}^{(n+1)}\left(z+\frac{1}{2}+\frac{1}{2}n\right)=B_{n+1+k}^{(n+1)}\left(-z+\frac{1}{2}+\frac{1}{2}n\right)$$

to get, by (2.4),

$$B_{n+k}^{(n)}\left(z+\frac{1}{2}+\frac{1}{2}n\right)=-B_{n+k}^{(n)}\left(-z-1+\frac{1}{2}+\frac{1}{2}n\right).$$

Letting $y=z+\frac{1}{2}$, we obtain, for $n+k$ odd:

$$B_{n+k}^{(n)}\left(y+\frac{1}{2}n\right)=-B_{n+k}^{(n)}\left(-y+\frac{1}{2}n\right).$$

This completes the proof.

We note that Theorem 3.1 implies, for $k \geq 0$ and $n \geq 0$,

$$B_{n+k}^{(n)}(n)=(-1)^{n+k}B_{n+k}^{(n)}. \tag{3.2}$$

Now, since $B_n^{(n)}(z+1)=n!\Psi_n(z)$, and since Jordan [9], p. 265, has shown

$$\frac{d}{dz}\Psi_n(z)=\binom{z}{n-1}=\frac{1}{(n-1)!}\sum_{r=0}^{n-1}s(n-1,r)z^r,$$

it follows that

$$B_{n+1}^{(n+1)}(z)=(n+1)\sum_{r=0}^{n}\frac{1}{r+1}s(n,r)(z-1)^{r+1}+B_{n+1}^{(n+1)}(1). \tag{3.3}$$

Equation (3.3) was also proved in [8], with different notation. Integrating (3.3) k times, using (2.3), we obtain

$$\begin{aligned}B_{n+1+k}^{(n+1)}(z)=&\binom{n+k+1}{k+1}\sum_{r=0}^{n}\binom{r+k+1}{r}^{-1}s(n,r)(z-1)^{r+k+1}\\&+\sum_{r=0}^{k}\binom{n+k+1}{r}B_{n+k+1-r}^{(n+1)}(1)(z-1)^r.\end{aligned} \tag{3.4}$$

We now plug $z=n+1$ into (3.4). By (2.17) and (2.5), the first two terms in the last summation are

$$B_{n+k+1}^{(n+1)}(1)=(n+k+1)B_{n+k}^{(n)}+B_{n+k+1}^{(n+1)},$$

$$(n+k+1)nB_{n+k}^{(n+1)}(1)=-k(n+k+1)B_{n+k}^{(n)},$$

so by (3.2) we have, if $n+k$ is odd:

$$\begin{aligned}(k-1)(n+k+1)B_{n+k}^{(n)}=&\binom{n+k+1}{k+1}\sum_{r=1}^{n}\binom{r+k+1}{r}^{-1}s(n,r)n^{r+k+1}\\&+\sum_{r=2}^{k}\binom{n+k+1}{r}B_{n+k+1-r}^{(n+1)}(1)n^r,\end{aligned} \tag{3.5}$$

and if $n+k$ is even, we have

$$\begin{aligned}-2B_{n+k+1}^{(n+1)}=&\binom{n+k+1}{k+1}\sum_{r=1}^{n}\binom{r+k+1}{r}^{-1}s(n,r)n^{r+k+1}\\&+\sum_{r=2}^{k}\binom{n+k+1}{r}B_{n+k+1-r}^{(n+1)}(1)n^r+(n+k+1)(1-k)B_{n+k}^{(n)}.\end{aligned} \tag{3.6}$$

It is important to remember that (3.5) is valid when $n+k$ is odd, and (3.6) is valid when $n+k$ is even. We are now in a position to prove congruences (1.6)-(1.9).

Theorem 3.2: If p is prime, $p>5$, then $B_{p+2}^{(p+1)} \equiv -\frac{1}{12}(p+2)!\,p^2 \pmod{p^6}$.

Proof: In (3.6), let $n=p$ and $k=1$. Then we have

$$B_{p+2}^{(p+1)} \equiv -\frac{1}{2}(p+2)(p+1)\sum_{r=1}^{4}\frac{s(p,r)}{(r+1)(r+2)}p^{r+2} \pmod{p^6}.$$

It is well known [5], pp. 218 and 229, that

$$s(p,j) \equiv 0 \pmod{p} \quad (1<j<p),$$
$$s(p,2j) \equiv 0 \pmod{p^2} \quad \left\{1 \le j \le \frac{1}{2}(p-3)\right\},$$

so we have $B_{p+2}^{(p+1)} \equiv -\frac{1}{12}(p+2)(p+1)s(p,1)p^3 \equiv -\frac{1}{12}(p+2)(p+1)(p-1)!\,p^3 \pmod{p^6}$. This completes the proof.

Theorem 3.2 extends Carlitz's congruence (1.3) and the work of Olson [11]. The motivation for (1.3) was evidently the congruence $B_{p+2}^{(p+1)} \equiv 0 \pmod{p^2}$, which was proved by S. Wachs [12] in 1947.

We will return to (3.5) later to prove congruences for $B_{p+2}^{(p)}$ and $B_{p+4}^{(p)}$. Next we prove two recurrence formulas that will be useful. Both formulas are given in [10], p. 150, for $k=0$ only.

Theorem 3.3: For $k \ge 0$,

$$\frac{B_n^{(n-k+1)}}{n!} = \sum_{r=0}^{n}\frac{(-1)^{n-r}}{n+1-r}\,\frac{B_r^{(r-k)}}{r!}.$$

Proof: In the first equation of (2.2), we replace n by $n+1$, we replace k by $n+1-k$, and we let $y=0$. We then subtract $B_{n+1}^{(n+1-k)}$ from both sides and divide by x to obtain

$$\frac{B_{n+1}^{(n+1-k)}(x)-B_{n+1}^{(n+1-k)}}{x} = \sum_{j=1}^{n+1}\binom{n+1}{j}(x-1)(x-2)\cdots(x-j+1)B_{n+1-j}^{(n+1-k-j)}. \tag{3.7}$$

We now take the limit as $x \to 0$ of both sides of (3.7). The limit of the left side is

$$\lim_{x\to 0}\frac{d}{dx}B_{n+1}^{(n+1-k)}(x) = (n+1)B_n^{(n+1-k)}.$$

Thus, we have

$$(n+1)B_n^{(n+1-k)} = \sum_{j=1}^{n+1}\binom{n+1}{j}(-1)^{j-1}(j-1)!\,B_{n+1-j}^{(n+1-k-j)},$$

and Theorem 3.3 follows by dividing both sides by $(n+1)!$ and letting $r=n+1-j$. This completes the proof.

Theorem 3.4: For $k \ge 0$,

$$\frac{(-1)^{n+k}B_{n+k}^{(n)}}{(n+k)!} = \sum_{r=0}^{n}\binom{n}{r}\frac{B_{r+k}^{(r)}}{(r+k)!}.$$

Proof: In the first equation of (2.2), replace n by $n+k$, replace k by n, let $y=0$, and let $x=n$. Theorem 3.4 now follows from (3.2), and the proof is complete.

Now for $k \geq 0$, p prime, and $[x]$ the greatest integer function, define

$$A_k(p; n) = \frac{(-1)^n p^{[n/(p-1)]}}{n!} B_n^{(n-k)}. \tag{3.8}$$

It was proved in [8] that $A_0(p; n)$ is integral (mod p); we now show that $A_k(p; n)$ has that same property. We note that $A_k(p; 0) = 1$, by (2.9). Theorem 3.3 gives us

$$A_k(p; n) = A_{k-1}(p; n) - \sum_{r=0}^{n-1} \frac{p^{[n/(p-1)]-[r/(p-1)]}}{n+1-r} A_k(p; r). \tag{3.9}$$

It was proved in [8] that if $p^t | (n+1-r)$ then $[n/(p-1)]-[r/(p-1)] \geq t$. Therefore, we can use induction on k and on n in (3.9) to prove $A_k(p; n)$ is integral (mod p). In fact, it follows from (2.5) that

$$\frac{1}{n-k} A_k(p; n) = -\frac{1}{k} A_{k-1}(p; n) + \frac{1}{k} A_{k-1}(p; n-1) p^{[n/(p-1)]-[(n-1)/(p-1)]},$$

so if p does not divide k, we see that $\frac{1}{n-k} A_k(p; n)$ is integral (mod p). Before putting this information together in a theorem, we make the following definitions.

Let $\alpha_p(n; k)$ denote the exponent of the highest power of p dividing the denominator of $B_n^{(n-k)}$ and let $\upsilon_p(n)$ denote the exponent of the highest power of p dividing $n!$ It is well known that if

$$n = n_0 + n_1 p + n_2 p^2 + \cdots + n_m p^m \quad (0 \leq n_i < p), \tag{3.10}$$

then $\upsilon_p(n) = \frac{1}{p-1}(n - n_0 - n_1 - \cdots - n_m)$.

We can now state the following theorem.

Theorem 3.5: Let p be prime and let $k \geq 0$. Let n have base p expansion (3.10) and let $\alpha_p(n; k)$ and $\upsilon_p(n)$ be as defined above. Then

$$\alpha_p(n; k) \leq \left[\frac{n}{p-1}\right] - \upsilon_p(n) = \left[\frac{n_0 + n_1 + \cdots + n_m}{p-1}\right].$$

If $p^j | (n-k)$ and p does not divide k, then

$$\alpha_p(n; k) \leq \left[\frac{n_0 + n_1 + \cdots + n_m}{p-1}\right] - j.$$

Corollary: Suppose n has base p expansion (3.10) and suppose $n_0 + n_1 + \cdots + n_m < p-1$. If $p^j | (n-k)$ and p does not divide k, then $B_n^{(n-k)} \equiv 0 \pmod{p^j}$. For example if $0 < k < p-2$ and $j \geq 1$, then $B_{p^j+k}^{(p^j)} \equiv 0 \pmod{p^j}$.

Theorem 3.6: Let $A_k(p; n)$ be defined by (3.8). Then, for $h \geq 0$ and p prime,

$$A_k(p; h(p-1)) \equiv (-1)^h \binom{h+k}{k} \pmod{p}, \tag{3.11}$$

$$A_k(p; h(p-1)+1) \equiv \frac{1}{2}(-1)^{h-1}(h+k-1)\binom{h+k}{k} \pmod{p} \quad (p>2). \tag{3.12}$$

Proof: We will use equation (3.9). It was proved in [8] that we can have

$$p^w | (n+1-r) \text{ and } \left[\frac{n}{p-1}\right] - \left[\frac{r}{p-1}\right] = w$$

only when $w=0$ or $w=1$. Thus, we have, for $0 \leq t < p-1$,

$$\begin{aligned} A_k(p; h(p-1)+t) &\equiv A_{k-1}(p; h(p-1)+t) - A_k(p; (h-1)(p-1)+t) \\ &\quad - \sum_{i=0}^{t-1} \frac{1}{t-i+1} A_k(p; h(p-1)+i) \pmod{p}. \end{aligned}$$

In particular, for $t=0$, we have

$$A_k(p; h(p-1)) \equiv A_{k-1}(p; h(p-1)) - A_k(p; (h-1)(p-1)) \pmod{p}. \tag{3.13}$$

In [8] it was proved that (3.11) is true for $k=0$. Also, $A_k(p; 0) = 1$. Thus, we can use induction on k and on h in (3.13) to prove that (3.11) is true for all k and h.

To prove (3.12), we first note that Theorem 3.4 tells us that if $n+k$ is odd, then

$$2A_k(p; n+k) = \sum_{r=0}^{n-1} \binom{n}{r} (-1)^{r+k} A_k(p; r+k) p^{[(n+k)/(p-1)]-[(r+k)/(p-1)]}.$$

Thus,

$$\begin{aligned} 2A_k(p; h(p-1)+1) &\equiv (h(p-1)+1-k) A_k(p; h(p-1)) \\ &\equiv -(h+k-1)(-1)^h \binom{h+k}{k} \pmod{p}, \end{aligned}$$

and the proof is complete.

For certain values of n, Theorem 3.6 gives us the exact value of $\alpha_p(n; k)$. For example, suppose $p=2$ and

$$\begin{aligned} n &= n_0 + n_1 2 + n_2 2^2 + \cdots + n_m 2^m \quad (0 \leq n_i \leq 1), \\ n+k &= t_0 + t_1 2 + t_2 2^2 + \cdots + t_m 2^m \quad (0 \leq t_i \leq 1), \\ k &= k_0 + k_1 2 + k_2 2^2 + \cdots + k_m t^m \quad (0 \leq k_i \leq 1). \end{aligned}$$

By Theorem 3.6, we see that if $k_i \leq t_i$ for all i, then

$$\alpha_2(n; k) = n - \upsilon_2(n) = n_0 + n_1 + \cdots + n_m. \tag{3.14}$$

In particular, if $n = 2^j$, then $\alpha_2(n; k) = 1$ for all $k \neq n$; that is, if n is a power of 2, then 2, but not 4, divides the denominator of $B_n^{(n-k)}$ for all k such that $0 \leq k < n$. More generally, if $2^j | n$ and $k < 2^j$, then (3.14) holds.

Theorem 3.7: If $p > 5$, we have

$$B_{p+2}^{(p)} \equiv \frac{1}{24} p^2 (p+2)!(p+12b_{p+1}) \pmod{p^7}, \tag{3.15}$$

$$B_{p+4}^{(p)} \equiv \frac{1}{12} p^2 (p+4)!(3p+2)b_{p+3} \pmod{p^4}, \tag{3.16}$$

where b_n is the Bernoulli number of the second kind, defined by (2.11). In general,

$$B_{p+2k}^{(p)} \equiv 0 \pmod{p^2} \quad \left\{k = 1, 2, \ldots, \frac{1}{2}(p-3)\right\}.$$

Proof: In (3.5), let $n = p$ and let $k = 2$. Then we have

$$\begin{aligned} B_{p+2}^{(p)} &\equiv \frac{1}{24}(p+2)(p+1)s(p,1)p^4 + \frac{1}{2}(p+2)B_{p+1}^{(p+1)}(1)p^2 \\ &\equiv \frac{1}{24}(p+2)!\,p^3 + \frac{1}{2}(p+2)(p+1)!\,b_{p+1}p^2 \pmod{p^7}, \end{aligned}$$

and (3.15) is proved. Now in (3.5) we let $n = p$ and $k = 4$ to get

$$3(p+5)B_{p+4}^{(p)} \equiv \sum_{r=2}^{4} \binom{p+5}{r} B_{p+5-r}^{(p+1)}(1)p^r \pmod{p^4}.$$

By (2.17), Theorem 3.2, and (3.15), we see that

$$p^3 B_{p+2}^{(p+1)}(1) \equiv 0 \equiv p^4 B_{p+1}^{(p+1)}(1) \pmod{p^4}.$$

Thus, we have

$$B_{p+4}^{(p)} \equiv \frac{1}{6}(p+4)B_{p+3}^{(p+1)}(1)p^2 \pmod{p^4}. \tag{3.17}$$

By (2.5), (2.12), and Theorem 3.2,

$$B_{p+3}^{(p+1)}(1) \equiv -\frac{1}{2}(p+1)B_{p+3}^{(p+2)} \equiv \frac{1}{2}(p+1)(p+2)(p+3)!\,b_{p+3} \pmod{p^2}, \tag{3.18}$$

and we know $(p+3)!\,b_3$ is integral (mod p) by (2.12). The proof of (3.16) now follows immediately from (3.17) and (3.18). The last statement of Theorem 3.7 is clear from (3.5) and the proof is complete.

We next derive another formula like (3.4). By (2.7) we have

$$\frac{d}{dz} B_n^{(n)}(z) = n\sum_{r=1}^{n} s(n,r)z^{r-1}, \quad \text{so} \quad B_n^{(n)} = n\sum_{r=1}^{n} \frac{1}{r} s(n,r)z^r + B_n^{(n)}.$$

Integrating k times, using (2.3), we get

$$B_n^{(n-k)}(z) = \binom{n}{k+1} \sum_{r=1}^{n-k} \binom{r+k}{k+1}^{-1} s(n-k,r)z^{r+k} + \sum_{j=0}^{k} \binom{n}{j} B_{n-j}^{(n-k)} z^j. \tag{3.19}$$

Equation (3.19) also follows directly from the second equality of (2.2). By (3.2) and (3.19) we have, for $n+k$ odd,

$$-2B_{n+k}^{(n)}=\binom{n+k}{k+1}\sum_{r=1}^{n}\binom{r+k}{k+1}^{-1}s(n,r)n^{r+k}+\sum_{j=1}^{k}\binom{n+k}{j}B_{n+k-j}^{(n)}n^{j}. \tag{3.20}$$

Carlitz [4] proved that $B_m^{(p)}$ is integral (mod p), $p\geq 3$, unless $m\equiv 0 \pmod{p-1}$ and $m\equiv 0$ or $p-1 \pmod{p}$, in which case $pB_m^{(p)}$ is integral. We note that by (3.20), with $n+k=m$ and $n=p$, we can say: If m is odd, if $p|m$, and if $p-1$ does not divide $m-1$, then $B_m^{(p)}\equiv 0 \pmod{p^2}$.

4. THE NUMBERS $B_n^{(n-1)}$

Because of their close relationship to the Bernoulli numbers of the second kind, that is, $B_n^{(n-1)}=(1-n)n!b_n$ (proved in section 2), the numbers $B_n^{(n-1)}$ deserve special consideration. We first note that, by (2.15) and (2.16), we have the generating function

$$\frac{x^2}{(1+x)\{\log(1+x)\}^2}=\sum_{n=0}^{\infty}B_n^{(n-1)}\frac{x^n}{n!}.$$

If we integrate the right side of (2.8) we have, for $n\geq 0$,

$$B_n^{(n-1)}=(1-n)\sum_{r=1}^{n}\frac{1}{r+1}s(n,r), \tag{4.1}$$

which provides a way of computing $B_n^{(n-1)}$ if a table of Stirling numbers is available. For example,

$$B_3^{(2)}=-2\left\{\frac{1}{2}s(3,1)+\frac{1}{3}s(3,2)+\frac{1}{4}s(3,3)\right\}=-2\left(\frac{1}{2}\cdot 2-\frac{1}{3}\cdot 3+\frac{1}{4}\right)=-\frac{1}{2}.$$

Equation (4.1) was also given in [9], p. 267, as a formula for b_n.

Another useful formula is the following: If n is odd, then

$$B_{n+2}^{(n+1)}=\binom{n+2}{2}\sum_{r=0}^{n+1}\frac{1}{r+1}s(n+1,r)n^{r+1}. \tag{4.2}$$

Equation (4.2) follows from [9], p. 267,

$$(n+1)!\Psi_{n+2}(z)=\sum_{r=0}^{n+1}\frac{1}{r+1}s(n+1,r)z^{r+1}+(n+1)!b_{n+2}, \tag{4.3}$$

where $\Psi_n(z)$ is defined by (2.13). If we plug $z=n$ into (4.3) and use $\Psi_{n+2}(n)=(-1)^n b_{n+2}$, which follows from (2.14), then (4.2) follows for odd n. We can now prove the following theorem.

Theorem 4.1: If n is odd and composite, $n>9$, then $B_{n+2}^{(n+1)}\equiv 0 \pmod{n^4}$.

Proof: It was proved in [8] that if $r\geq 3$ and n is odd and composite, $n>9$, then $\frac{1}{r+1}n^{r+1}\equiv 0$ (mod n^4). Thus, by (4.2), we have

$$B_{n+2}^{(n+1)}\equiv\binom{n+2}{2}\left\{\frac{1}{2}s(n+1,1)n^2+\frac{1}{3}s(n+1,2)n^3\right\} \pmod{n^4}. \tag{4.4}$$

Now for n composite and $n>9$ (see [6], p. 217),

$$s(n+1,1) = -n! \equiv 0 \pmod{n^2},$$
$$s(n+1,2) = n!\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right) \equiv 0 \pmod{n}.$$

Also, we can easily see that if $3^j | n$, then

$$s(n+1,2) \equiv 0 \pmod{3^{j+1}} \quad (j>2).$$

Thus, Theorem 4.1 follows from (4.4), and the proof is complete.

For convenience, we again use the notation

$$A_1(p;n) = \frac{(-1)^n p^{[n/(p-1)]}}{n!} B_n^{(n-1)}.$$

Because of (2.12), many properties of b_n and $B_n^{(n-1)}$ follow from properties of $B_n^{(n)}$. Using results in [8], we can write down the following:

$$\frac{1}{1-n} A_1(2;n) \equiv 1 \pmod 8 \quad (n \neq 1), \tag{4.5}$$

$$\frac{1}{2r-1} A_1(3;2r) \equiv (-1)^{r-1}(3r^3+3r+1) \pmod 9, \tag{4.6}$$

$$\frac{1}{2r} A_1(3;2r+1) \equiv (-1)^{r-1}(4r^3+3r^2+1) \pmod 9 \quad (r \geq 1). \tag{4.7}$$

Congruence (4.5) gives us $\alpha_2(n;1)$, the exact power of 2 dividing the denominator of $B_n^{(n-1)}$. Using the notation of section 3, we have $\alpha_2(n;1) = n - \upsilon_2(n) - j = n_0 + n_1 + \cdots + n_m - j$, where 2^j is the highest power of 2 dividing $n-1$, and $n_0, n_1, \ldots, n_m$ are the digits in the base 2 expansion of n. Similarly, if n is not an odd integer congruent to 2 (mod 3), then (4.6) and (4.7) give

$$\alpha_3(n,1) = \left[\frac{n}{2}\right] - \upsilon_3(n) - j = \left[\frac{n_0+n_1+\cdots+n_m}{2}\right] - j. \tag{4.8}$$

where 3^j is the highest power of 3 dividing $n-1$, and $n_0, n_1, \ldots, n_m$ are the digits in the base 3 expansion of n. If n is an odd integer congruent to 2 (mod 3), we must replace the first "equals" symbol in (4.8) by "<."

We know from section 3 that $\frac{1}{1-n} A_1(p;n)$ is integral (mod p) for any $n \neq 1$.

Jordan [9], p. 267, proved $(-1)^{n+1} b_n > 0$ for $n > 0$. Hence, we have $(-1)^n B_n^{(n-1)} > 0$ $(n > 1)$. In general, the sign of $B_n^{(n-k)}$ is not known. It seems that the signs usually alternate when $n-k>0$, but there are exceptions. For example, $B_{16}^{(10)}$ and $B_{17}^{(11)}$ are both positive, $B_8^{(3)}$ and $B_9^{(4)}$ are both positive, $B_{10}^{(7)}$ and $B_{11}^{(8)}$ are both negative.

Nörlund [10], p. 461, gave a table of values for $B_n^{(n-1)}$ for $n = 2, 3, \ldots, 12$, and Jordan [9], p. 266, listed b_n for $n = 0, 1, 2, \ldots, 10$. We give here the first fifteen values of $B_n^{(n-1)}$ with numerators and denominators factored.

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Table of the Numbers $B_n^{(n-1)}$

$B_0^{(-1)} = 1$	
$B_1^{(0)} = 0$	$B_8^{(7)} = \frac{7 \cdot 19 \cdot 1787}{2 \cdot 3^2 \cdot 5}$
$B_2^{(1)} = \frac{1}{2 \cdot 3}$	$B_9^{(8)} = -\frac{2 \cdot 7^3 \cdot 167}{5}$
$B_3^{(2)} = -\frac{1}{2}$	$B_{10}^{(9)} = \frac{3 \cdot 3250433}{2^2 \cdot 11}$
$B_4^{(3)} = \frac{19}{2 \cdot 5}$	$B_{11}^{(10)} = -\frac{3^7 \cdot 5^2 \cdot 173}{2^2}$
$B_5^{(4)} = -9$	$B_{12}^{(11)} = \frac{11 \cdot 541 \cdot 4801 \cdot 5273}{2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 13}$
$B_6^{(5)} = \frac{5 \cdot 863}{2^2 \cdot 3 \cdot 7}$	$B_{13}^{(12)} = -\frac{11^3 \cdot 2207 \cdot 8329}{2 \cdot 5 \cdot 7}$
$B_7^{(6)} = -\frac{5^3 \cdot 11}{2^2}$	$B_{14}^{(13)} = \frac{13 \cdot 132282840127}{2^3 \cdot 3^2 \cdot 5}$

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