ADVANCED PROBLEMS AND SOLUTIONS

Edited by Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-490 ***Proposed by A. Stuparu, Valcea, Romania (corrected)***

Prove that the equation $S(x)=p$, where p is a given prime number, has just $D((p-1)!)$ solutions, all of them in between p and $p!$ [$S(n)$ is the Smarandache Function: the smallest integer such that $S(n)!$ is divisible by n, and $D(n)$ is the number of positive divisors of n.]

H-496 ***Proposed by Paul S. Bruckman, Edmonds, WA***

Let n be a positive integer >1 with $\gcd(n, 10)=1$, and $\delta=(5/n)$, a Jacobi symbol. Consider the following congruences:

(1) $F_{n-\delta}\equiv 0 \pmod n$, $L_n\equiv 1 \pmod n$;

(2) $F_{\frac{1}{2}(n-\delta)}\equiv 0 \pmod n$ if $n\equiv 1 \pmod 4$, $L_{\frac{1}{2}(n-\delta)}\equiv 0 \pmod n$ if $n\equiv 3 \pmod 4$.

Composite n which satisfy (1) are called *Fibonacci-Lucas pseudoprimes*, which is abbreviated as "FLUPPS." Composite n which satisfy (2) are called *Euler-Lucas pseudoprimes with parameters (1, –1)*, abbreviated as "ELUPPS." Prove that FLUPPS and ELUPPS are equivalent.

H-497 ***Proposed by Mohammad K. Azarian, University of Evansville, Evansville, IN***

Solve the recurrence relation

$$\sum_{i=0}^{k}\left(\prod_{j=0}^{k}\frac{x_{n-j}}{x_{n-i}}\right)^r+\left(\prod_{t=0}^{k}x_{n-t}\right)^r=0,$$

where r is any nonzero real number, $n>k\geq 1$, and $x_m\neq 0$ for all m.

H-498 ***Proposed by Paul S. Bruckman, Edmonds, WA***

Let $u=u_e=L_{2^e}$, $e=2, 3, \ldots$. Show that if u is composite it is both a Fibonacci pseudoprime (or "FPP") and a Lucas pseudoprime (or "LPP"). Specifically, show that $u\equiv 7 \pmod{10}$, $F_{u+1}\equiv 0 \pmod u$, and $L_u\equiv 1 \pmod u$.

SOLUTIONS

Quite Prime

H-483 ***Proposed by James Nicholas Boots (deceased) & Lawrence Somer, The Catholic University of America, Washington, D.C.***
(Vol. 32, no. 1, February 1994)

Let $m \geq 2$ be an integer such that

$$L_m \equiv 1 \pmod{m}. \tag{1}$$

It is well known (see [1], p. 44) that if m is a prime, then (1) holds. It has been proved by H. J. A. Duparc [3] that there exist infinitely many composite integers, called Fibonacci pseudoprimes, such that (1) holds. It has also been proved in [2] and [4] that every Fibonacci pseudoprime is odd.

(i) Prove that $L_{m-1}^2 + L_{m-1} - 6 \equiv 0 \pmod{m}$.

In particular, conclude that if m is prime, then $L_{m-1} \equiv 2$ or $-3 \pmod{m}$.

(ii) Prove that $F_{m-2} - L_{m-1}F_{m-1} \equiv 1 \pmod{m}$.

References

1. R. D. Carmichael. "On the Numerical Factors of the Arithmetic Forms $\alpha^n \pm \beta^n$." *Ann. Math.* Second Series **15** (1913):30-70.
2. A. Di Porto. "Nonexistence of Even Fibonacci Pseudoprimes of the 1st Kind." *The Fibonacci Quarterly* **31.2** (1993):173-77.
3. H. J. A. Duparc. *On Almost Primes of the Second Order*, pp. 1-13. Amsterdam: Rapport ZW, 1955-013, Math. Center, 1955.
4. D. J. White, J. N. Hunt, & L. A. G. Dresel. "Uniform Huffman Sequences Do Not Exist." *Bull. London Math. Soc.* **9** (1977):193-98.

Solution by the Proposer

(i) If $m = 2$, then

$$L_{m-1}^2 + L_{m-1} - 6 = L_1^2 + L_1 - 6 = 1^2 + 1 - 6 = -4 \equiv 0 \pmod{2}$$

and

$$L_{m-1} = L_1 = 1 \equiv -3 \pmod{2}.$$

Now assume that $m > 2$. Then m is odd. It is well-known that

$$L_{2n} = L_n^2 - 2(-1)^n. \tag{2}$$

Thus,

$$L_{2m} = L_m^2 - 2(-1)^m \equiv 1^2 - 2(-1) \equiv 3 \pmod{m}. \tag{3}$$

Further, it follows by identity (I_{31}) on page 59 of *Fibonacci and Lucas Numbers* by Verner E. Hoggatt, Jr., that

$$L_{2m-1} = L_m L_{m-1} - (-1)^{m-1} \equiv (1)L_{m-1} - 1 \equiv L_{m-1} - 1. \tag{4}$$

By (2),

$$L_{2m-2} = L_{m-1}^2 - 2(-1)^{m-1} \equiv L_{m-1}^2 - 2(1) \equiv L_{m-1}^2 - 2 \pmod{m}. \tag{5}$$

Since $L_{2m} = L_{2m-1} + L_{2m-2}$, it follows by (3), (4), and (5) that

$$3 \equiv L_{m-1} - 1 + L_{m-1}^2 - 2 \pmod{m}, \tag{6}$$

which implies that

$$L_{m-1}^2 + L_{m-1} - 6 \equiv 0 \pmod{m}. \tag{7}$$

Since

$$L_{m-1}^2 + L_{m-1} - 6 = (L_{m-1} - 2)(L_{m-1} + 3), \tag{8}$$

it follows from (7) and (8) that $L_{m-1} \equiv 2$ or $-3 \pmod{m}$ if m is prime.

(ii) If $m = 2$, then

$$F_{m-2} - L_{m-1}F_{m-1} = F_0 - L_1F_1 = 0 - (1)(1) = -1 \equiv 1 \pmod{2}.$$

Now assume that $m > 2$. Then m is odd. We will first prove by induction that

$$L_{m-k} \equiv (-1)^k (F_{k-1} - L_{m-1}F_k) \pmod{m} \tag{9}$$

for $k \geq 0$. If $k = 0$, then

$$L_{m-k} = L_m \equiv 1 \equiv (-1)^0 (F_{-1} - L_{m-1}F_0) \equiv (1)(1 - L_m \cdot 0) \equiv 1 \pmod{m}.$$

If $k = 1$, then

$$L_{m-k} = L_{m-1} \equiv (-1)^1 (F_0 - L_{m-1}F_1) \equiv (-1)(0 - L_{m-1}(1)) \equiv L_{m-1} \pmod{m}.$$

Now assume that (9) holds up to $k = r$. Then

$$\begin{aligned} L_{m-(r+1)} &= L_{m-(r-1)} - L_{m-r} \\ &\equiv (-1)^{m-(r-1)}(F_{r-2} - L_{m-1}F_{r-1}) - (-1)^{m-r}(F_{r-1} - L_{m-1}F_r) \\ &\equiv (-1)^{m-(r+1)}((F_{r-2} + F_{r-1}) - L_{m-1}(F_{r-1} + F_r)) \\ &\equiv (-1)^{m-(r+1)}(F_r - L_{m-1}F_{r+1}) \pmod{m}. \end{aligned}$$

Thus, (9) holds for $k \geq 0$. Now let $k = m - 1$. Since m is odd, it follows by (9) that

$$L_{m-(m-1)} = L_1 = 1 \equiv (-1)^{m-1}(F_{m-2} - L_{m-1}F_{m-1}) \equiv F_{m-2} - L_{m-1}F_{m-1} \pmod{m}.$$

Also solved by P. Bruckman, L. Dresel, and H. Seiffert.

Strictly Monotone

H-484 ***Proposed by J. Rodriguez, Sonora, Mexico***
(Vol. 32, no. 1, February 1994)

Find a strictly increasing infinite series of integer numbers such that, for any consecutive three of them, the Smarandache Function is neither increasing nor decreasing.

*Find the largest strictly increasing series of integer numbers for which the Smarandache Function is strictly decreasing.

Solution by Paul S. Bruckman, Edmonds, WA

Solution to Part 1: For a given natural n, the *Smarandache Function* of n, denoted by $S(n)$, is defined to be the smallest natural m such that $n \mid m!$.

The following results ensue from the definition:

$$S(n) = \max_{p^e \| n}\{S(p^e)\}; \tag{1}$$

$$S(p^e) = ep, \quad \text{if } p \geq e; \tag{2}$$

$$S(n!) = n. \tag{3}$$

Given m natural, we define $U(m)$ to be the set of natural n such that $S(n) = m$ for all $n \in U(m)$. Then $n \in U(m)$ iff $n | m!$ and $n \nmid (m-1)!$ We may easily show from this that

$$U(m) = \bigcup_{p|m,\ p^e \| (m-1)!} \{p^{e+1}d : d \mid p^{-e-1} \cdot m!\} \tag{4}$$

In particular, if m is equal to p, a prime,

$$U(p) = \{pd : d \mid (p-1)!\}. \tag{5}$$

For example, $U(2) = \{2\}$, $U(3) = \{3, 6\}$, $U(5) = \{5, 10, 15, 20, 30, 40, 60, 120\}$, etc.

Thus, the smallest element of $U(p)$ is p, while the largest is $p!$. The number of elements of $U(p)$ is $\tau((p-1)!)$, which increases rapidly with increasing p.

Using these facts, we may construct an infinite sequence $X = \{x_n\}_{n\geq 1}$ with the properties required in part 1 of the problem. Incidentally, the wording of the problem, in both parts, should be changed to substitute the word "sequence" for "series."

We let $\{p_n\}_{n\geq 1} = \{2, 3, 5, 7, \ldots\}$ denote the sequence of primes. Our first step is to define an infinite sequence $E = \{e_n\}_{n\geq 1}$ of positive integers as follows:

$$e_{4u} = 2u+2,\ u = 1, 2, \ldots;\ e_{4u+1} = 2u+1,\ e_{4u+2} = 2u+3,\ e_{4u+3} = 2u+2,\ u = 0, 1, \ldots. \tag{6}$$

Thus, $E = \{1, 3, 2, 4, 3, 5, 4, 6, 5, 7, 6, 8, 7, 9, 8, 10, \ldots\}$.

Next, we define the sequence of primes Q as follows:

$$Q = \{p_{e_n}\}_{n\geq 1}. \tag{7}$$

Thus, $Q = \{2, 5, 3, 7, 5, 11, 7, 13, 11, 17, 13, 19, 17, 23, 19, 29, \ldots\}$.

Each distinct value of terms in E and Q occurs exactly twice, except the first and third values, which occur only once. Observe that no three consecutive terms of E are increasing or decreasing, since the values alternate in magnitude; the same is true of Q, since the primes form an increasing sequence.

We now set each term p_{e_n} of Q equal to $S(x_n)$ and seek to find x_n such that $X = \{x_n\}_{n\geq 1}$ is an increasing sequence of positive integers. For definiteness, we define x_n to be the *smallest* positive integer such that $x_n > x_{n-1}$, beginning with $x_1 = 2$. Using the result in (5), we may thus uniquely determine $x_n \in S^{-1}(Q)$ such that $x_n > x_{n-1}$, with $x_1 = 2$. We may illustrate by displaying the first 20 terms of X in the table below. Note that x_n is a multiple of p_{e_n} in all cases; indeed x_n is the smallest multiple of p_{e_n} satisfying the requirement that X is an increasing sequence. The process may be continued ad infinitum, yielding X, a solution to part 1 of the problem.

n	e_n	$p_{e_n} = S(x_n)$	x_n	n	e_n	$p_{e_n} = S(x_n)$	x_n
1	1	2	2	11	6	13	39
2	3	5	5	12	8	19	57
3	2	3	6	13	7	17	68
4	4	7	7	14	9	23	69
5	3	5	10	15	8	19	76
6	5	11	11	16	10	29	87
7	4	7	14	17	9	23	92
8	6	13	26	18	11	31	93
9	5	11	33	19	10	29	116
10	7	17	34	20	12	37	148

Solution to Part 2: Using the fact that $p|\binom{p}{n}$ for all $n \in \{1, 2, \ldots, p-1\}$, where p is prime, we see that $S(\binom{p}{n}) = p$ for these values. Moreover,

$$\binom{p}{1} < \binom{p}{2} < \cdots < \binom{p}{\frac{1}{2}(p-1)}, \; p \geq 5.$$

These facts enable us to construct a strictly increasing sequence of natural numbers, beginning with an arbitrary prime, for which the Smarandache Function is strictly decreasing.

Let $\{p_n\}_{n\geq 1} = \{2, 3, 5, \ldots\}$ denote the sequence of primes. Given $n > 1$, we may construct a sequence of binomial coefficients

$$V(p_n) = \left\{\binom{p_n}{m_1}, \binom{p_{n-1}}{m_2}, \ldots, \binom{p_{n-r+1}}{m_r}\right\},$$

where the m_i's are chosen to be the minimum natural numbers subject to $1 = m_1 < m_2 < \cdots < m_r \leq \frac{1}{2}(p_{n-r+1} - 1)$, such that

$$\binom{p_n}{m_1} < \binom{p_{n-1}}{m_2} < \cdots < \binom{p_{n-r+1}}{m_r}.$$

We may choose $m_i = i$ for $i \leq s$, say, but require $m_i > i$ for all $i > s$. The number of terms in the sequence, namely the integer r, depends solely on n. The sequence $V(p_n)$ is finite because, for some r,

$$\binom{p_{n-r+2}}{m} < \binom{p_{n-r+1}}{m_r}$$

for all m. Note that

$$S\left(\binom{p_n}{m_1}\right) = p_n, \; S\left(\binom{p_{n-1}}{m_2}\right) = p_{n-1}, \ldots, S\left(\binom{p_{n-r+1}}{m_r}\right) = p_{n-r+1};$$

thus, $S(V(p_n))$ is a strictly decreasing sequence, as required.

We illustrate with two examples. If $n = 26$, we take $p_n = 101$. We may then take

$$\begin{aligned} V(101) &= \left\{\binom{101}{1}, \binom{97}{2}, \binom{89}{3}, \binom{83}{4}, \binom{79}{5}, \binom{73}{6}, \binom{71}{7}, \binom{67}{8}, \binom{61}{9}, \binom{59}{10}, \binom{53}{11}, \binom{47}{13}, \binom{43}{15}, \binom{41}{17}\right\} \\ &= \{x_n\}_{n=1}^{14}, \end{aligned}$$

say. We easily check that $x_1 < x_2 < x_3 < \cdots < x_{14}$, however, $101 > 97 > \cdots > 41$, i.e., $S(x_1) > S(x_2) > \cdots > S(x_{14})$.

For our second example, we take $n = 51$ (hence, $p_n = 233$). In this case, we take

$$V(233) = \left\{\binom{233}{1}, \binom{229}{2}, \binom{227}{3}, \binom{223}{4}, \binom{211}{5}, \binom{199}{6}, \binom{197}{7}, \binom{193}{8}, \binom{191}{9}, \binom{181}{10}, \right.$$
$$\binom{179}{11}, \binom{173}{12}, \binom{167}{13}, \binom{163}{14}, \binom{157}{15}, \binom{151}{16}, \binom{149}{17}, \binom{139}{18}, \binom{137}{19}, \binom{131}{20},$$
$$\left.\binom{127}{21}, \binom{113}{23}, \binom{109}{24}, \binom{107}{25}, \binom{103}{26}, \binom{101}{27}, \binom{97}{29}, \binom{89}{34}\right\}.$$

As we may verify, the sequence given above is an increasing sequence. The sequence terminates at the 28th term, since $\binom{83}{41} < \binom{89}{34}$.

Clearly, we may construct a sequence $V(p)$ in this fashion for all primes p of arbitrary size. The number of terms of $V(p)$ clearly grows with p in some fashion; apparently, $|V(p)| = 0(p/\log p)$ as $p \to \infty$, but this has not been established.

Also solved by H. Seiffert and the proposer.

Ghost from the Past

H-459 ***Proposed by Stanley Rabinowitz, Westford, MA (Vol. 29, no. 4, November 1991)***

Prove that, for all $n > 3$,

$$\frac{13\sqrt{5}-19}{10} L_{2n+1} + 4.4(-1)^n$$

is very close to the square of an integer.

Solution by H.-J. Seiffert, Berlin, Germany

We shall prove that

$$(5F_{n-1} - F_{n-3})^2 - \left(\frac{13\sqrt{5}-19}{10} L_{2n+1} + 4.4(-1)^n\right) = -2.6\sqrt{5}\beta^{2n+1}. \tag{1}$$

Since (β^{2n}) is a strictly decreasing sequence of positive reals, a simple calculation gives $0 < A_n \le 2.6(85-38\sqrt{5})$ for $n > 3$, where A_n denotes the left side of (1). Noting that $2.6(85-38\sqrt{5}) \sim 0.076492$, we see that the statement of the proposal is reasonable.

To prove (1), we use the following easily verifiable equations:

$$5F_{n-1}^2 = L_{2n-2} + 2(-1)^n;\; 5F_{n-1}F_{n-3} = L_{2n-4} + 3(-1)^n;$$
$$5F_{n-3}^2 = L_{2n-6} + 2(-1)^n = 3L_{2n-4} - L_{2n-2} + 2(-1)^n;\; L_{2n+1} = 5L_{2n-2} - 2L_{2n-4}.$$

Now, a straightforward calculation yields $10A_n = 13((11-5\sqrt{5})L_{2n-2} + 2(2-\sqrt{5})L_{2n-4})$ or, by $2-\sqrt{5} = \beta^3$ and $11-5\sqrt{5} = 2\beta^5$ and the Binet form of the Lucas numbers,

$$10A_n = 26(\beta^4 - 1)\beta^{2n-1} = 26(\beta^2 - \alpha^2)\beta^{2n+1} = -26\sqrt{5}\beta^{2n+1}.$$

This proves (1).

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