GENERATING FIBONACCI WORDS

Wai-fong Chuan*
Department of Mathematics, Chung-Yuan Christian University,
Chung-Li, Taiwan 320, Republic of China
(Submitted June 1993)

INTRODUCTION

A word w is called an n^{th}*-order Fibonacci word* derived from two distinct letters a and b if there exists a finite sequence $w_1, w_2, \ldots, w_n$ of words with $w_1 = a$, $w_2 = b$, $w_n = w$ and each w_k equals $w_{k-1}w_{k-2}$ or $w_{k-2}w_{k-1}$, $3 \le k \le n$. The basic structure of Fibonacci words has been studied in [2]. In this paper we discuss various methods of generating Fibonacci words.

Throughout this paper, let Q_n denote the set of all n^{th}-order Fibonacci words derived from distinct letters a and b. Some of these methods generate all the Fibonacci words in Q_n from any given u in Q_n without repetitions and some of them generate Q_n from Q_{n-1}.

1. BINARY TREES

Let $X = \{a, b\}$ be an alphabet of two letters and let X^* be the free monoid generated by X. Elements of X^* are called words. For any word $w = a_1a_2 \cdots a_n \in X^*$, define $f(w)$ [resp. $g(w)$] to be the word in X^* obtained by replacing each a in w by b and each b in w by ba (resp., by ab). Also define $T(w) = a_2 \cdots a_n a_1$ and $R(w) = a_n \cdots a_2 a_1$. A word w is called a *symmetric word* or a *palindrome* if $R(w) = w$.

Associated with each finite binary sequence $r_1, r_2, \ldots, r_{n-2}$ there are four words in X^*,

$$w_n^{r_1r_2\cdots r_{n-2}}, \quad w_n^{(r_1r_2\cdots r_{n-2})}, \quad w_n^{[r_1r_2\cdots r_{n-2}]}, \quad w_n^{\{r_1r_2\cdots r_{n-2}\}},$$

defined as follows:

$$w_1 = a, \qquad w_2 = b,$$

$$w_m^{r_1r_2\cdots r_{n-2}} = \begin{cases} w_{n-1}^{r_1r_2\cdots r_{n-3}} w_{n-2}^{r_1r_2\cdots r_{n-4}}, & \text{if } r_{n-2} = 0, \\ w_{n-2}^{r_1r_2\cdots r_{n-4}} w_{n-1}^{r_1r_2\cdots r_{n-3}}, & \text{if } r_{n-2} = 1; \end{cases}$$

$$w_n^{(r_1r_2\cdots r_{n-2})} = \begin{cases} R\left(w_{n-1}^{(r_1r_2\cdots r_{n-3})}\right) w_{n-2}^{(r_1r_2\cdots r_{n-4})}, & \text{if } r_{n-2} = 0, \\ w_{n-2}^{(r_1r_2\cdots r_{n-4})} R\left(w_{n-1}^{(r_1r_2\cdots r_{n-3})}\right), & \text{if } r_{n-2} = 1; \end{cases}$$

$$w_n^{[r_1r_2\cdots r_{n-2}]} = \begin{cases} f\left(w_{n-1}^{[r_1r_2\cdots r_{n-3}]}\right), & \text{if } r_{n-2} = 0, \\ g\left(w_{n-1}^{[r_1r_2\cdots r_{n-3}]}\right), & \text{if } r_{n-2} = 1; \end{cases}$$

and

* This research was supported in part by the National Science Council, R.O.C. Grant NSC 82-0208-M033-014.

$$w_n^{\{r_1r_2\cdots r_{n-2}\}} = \begin{cases} f\left(w_{n-1}^{\{r_1r_2\cdots r_{n-3}\}}\right), & \text{if either } r_{n-2}=0 \text{ and } n \text{ is odd} \\ & \text{or } r_{n-2}=1 \text{ and } n \text{ is even,} \\ g\left(w_{n-1}^{\{r_1r_2\cdots r_{n-3}\}}\right) & \text{if either } r_{n-2}=1 \text{ and } n \text{ is odd} \\ & \text{or } r_{n-2}=0 \text{ and } n \text{ is even,} \end{cases}$$

$n \geq 3$. The superscript does not appear if the subscript is less than or equal to 2. For simplicity, we denote $w_n^{00\ldots0}$ (resp. $w_n^{(00\ldots0)}$, $w_n^{[00\ldots0]}$, $w_n^{\{00\ldots0\}}$) by w_n^0 (resp. $w_n^{(0)}$, $w_n^{[0]}$, $w_n^{\{0\}}$).

The word $w_n^{r_1r_2\cdots r_{n-2}}$ [or, more precisely, $w_n^{r_1r_2\cdots r_{n-2}}(a,b)$] is an n^{th}-order Fibonacci word derived from the pair of initial letters (a, b). More generally, we can define n^{th}-order Fibonacci words derived from a pair of initial words (x, y) (see [2]).

Now we have four binary trees whose nodes are words. We shall prove in Theorem 1 that each level of these trees consists of the n^{th}-order Fibonacci words with repetitions. More precisely, the words in each level of each tree is just a permutation of the words of the same level of any other tree, with the number of repetitions of each word unchanged. The relations between the Fibonacci words $w_n^{r_1r_2\cdots r_{n-2}}$, $w_n^{(r_1r_2\cdots r_{n-2})}$, $w_n^{[r_1r_2\cdots r_{n-2}]}$, and $w_n^{\{r_1r_2\cdots r_{n-2}\}}$ tell us how a particular Fibonacci word can be generated in different ways.

Theorem 1: Let $n \geq 3$, $r_1, r_2, \cdots, r_{n-2}$ be a binary sequence and let $s_i = 1 - r_i$, $1 \leq i \leq n-2$. Then

(a) $R(w_n^{r_1r_2\cdots r_{n-2}}) = w_n^{s_1s_2\cdots s_{n-2}}$.

Similar results hold for $w_n^{(r_1r_2\cdots r_{n-2})}$, $w_n^{[r_1r_2\cdots r_{n-2}]}$, and $w_n^{\{r_1r_2\cdots r_{n-2}\}}$.

(b) $w_n^{[r_1r_2\cdots r_{n-2}]} = w_n^{r_{n-2}\cdots r_2r_1}$.

(c) $$w_n^{(r_1r_2\cdots r_{n-2})} = \begin{cases} w_n^{r_1s_2r_3\cdots s_{n-3}r_{n-2}} & (n \text{ odd}), \\ w_n^{s_1r_2\cdots s_{n-3}r_{n-2}} & (n \text{ even}). \end{cases}$$

(d) $$w_n^{\{r_1r_2\cdots r_{n-2}\}} = w_n^{(r_{n-2}\cdots r_2r_1)} = \begin{cases} w_n^{r_{n-2}s_{n-3}r_{n-4}\cdots s_2r_1} & (n \text{ odd}), \\ w_n^{s_{n-2}r_{n-3}\cdots s_2r_1} & (n \text{ even}), \end{cases}$$

$$= \begin{cases} w_n^{[r_1s_2r_3\cdots s_{n-3}r_{n-2}]} & (n \text{ odd}), \\ w_n^{[r_1s_2\cdots r_{n-3}s_{n-2}]} & (n \text{ even}). \end{cases}$$

Proof: First, note that part 1 of (a) has been proved in [2]. Part 3 (resp. part 2) of (a) follows from (b) [resp. (c)] and part 1 of (a).

Assertions (b), (c), and (d) are proved by induction.

We illustrate the theorem with the following examples.

Example 1: $\{w_n^0\}$ and $\{w_n^1\}$ are well-known sequences of Fibonacci words (see [4]). Recently they are used by Hendel and Monteferrante [6] and by Chuan [5] to solve an extraction problem of the golden sequence posed by Hofstadter [7].

By Theorem 1,

$$w_n^{[0]} = w_n^0 = \begin{cases} w_n^{(010\ldots10)} & (n \text{ odd}), \\ \mathrm{w}_\mathrm{n}^{(10\ldots10)} = R(w_n^{(010\ldots101)}) & (n \text{ even}), \end{cases} \quad (n \geq 3).$$

The first equality means that the sequence given by $w_1 = a$, $w_2 = b$, and $w_n = w_{n-1}w_{n-2}$ $(n \geq 3)$ is precisely the sequence $\{w_n\}$ where $w_1 = a$, $w_2 = b$, and w_n is obtained from w_{n-1} ;by replacing each a in w_{n-1} by b and each b in w_{n-1} by ba. The second equality means that, if $q_1 = a$, $q_2 = b$, and

$$q_n = \begin{cases} R(q_{n-1})q_{n-2} & (n \text{ odd}), \\ q_{n-2}R(q_{n-1}) & (n \text{ even}), \end{cases} \quad (n \geq 3),$$

then $w_n = q_n$ if n is odd and $w_n = R(q_n)$ if n is even. A similar result holds for w_n^1.

Example 2: By Theorem 1,

$$w_n^{[0101\ldots]} = \begin{cases} w_n^{010\ldots10} & (n \text{ odd}), \\ w_n^{10\ldots10} = R(w_n^{010\ldots101}) & (n \text{ even}), \end{cases}$$

$$= w_n^{(00\ldots0)} = T^{F_{n-1}-1}(w_n^0) = T(w_n^1).$$

See [2] for the last two equalities. Again, the sequence $\{w_n^{(0)}\}$ can be generated by three different methods. This is also observed by Anderson [1].

Example 3: Let $v_1 = a$, $v_2 = b$, and $v_n = v_{n-2}R(v_{n-1})$ $(n \geq 3)$. Then $v_n = R(w_n)$ where w_n is as in Example 2. This is because

$$v_n = w_n^{(11\ldots1)} = R(w_n^{(00\ldots0)}) = R(w_n).$$

Example 4: Let $w_1 = a$, $w_2 = b$, and $w_n = w_{n-1}R(w_{n-2})$ $(n \geq 3)$. Then

(a) $w_n = w_n^{r_1r_2\cdots r_{n-2}}$ where

$$r_i = \begin{cases} 1, & \text{if } i \equiv 0 \pmod 3, \\ 0, & \text{otherwise.} \end{cases}$$

(b) w_n is symmetric $\Leftrightarrow n \not\equiv 0 \pmod 3$; hence, $\{w_n\}$ contains all the symmetric Fibonacci words (see [3]).

(c) $w_{3k+2} = w_{3k+1}R(w_{3k}) = w_{3k}w_{3k+1}$, $k \geq 1$.

(d) $w_n = \begin{cases} R(w_{n-1})w_{n-2}, & \text{if } n \equiv 0 \pmod 3, \\ w_{n-2}R(w_{n-1}), & \text{otherwise.} \end{cases}$

(e) $w_n = w_n^{(t_1t_2\cdots t_{n-2})}$ where $t_i = \begin{cases} 0, & \text{if } i \equiv 1 \pmod 3, \\ 1, & \text{otherwise.} \end{cases}$

2. LOCATING THE LETTERS

For $n > 2$, let

$$s = \begin{cases} F_{n-1} & (n \text{ even}), \\ F_{n-2} & (n \text{ odd}); \end{cases}$$

and

$$t = \begin{cases} F_{n-2} & (n \text{ even}), \\ F_{n-1} & (n \text{ odd}). \end{cases}$$

Theorem 2: Let $n > 2$, $q_n = w_n^{10101\ldots}$, $T^{js}(q_n) = c_1 c_2 \cdots c_{F_n}$ where $c_i \in \{a, b\}$. Then

$$\begin{aligned} c_k = a &\Leftrightarrow k \equiv (r+j)t \pmod{F_n} && \text{for some } 1 \le r \le F_{n-2} \\ &\Leftrightarrow k \equiv (r-j)s \pmod{F_n} && \text{for some } F_{n-1} \le r \le F_n - 1 \\ &\Leftrightarrow k \equiv 1+(r-j)s \pmod{F_n} && \text{for some } 0 \le r \le F_{n-2} - 1 \\ &\Leftrightarrow k \equiv 1+(r+j)t \pmod{F_n} && \text{for some } F_{n-1}+1 \le r \le F_n. \end{aligned} \tag{1}$$

$$\begin{aligned} c_k = b &\Leftrightarrow k \equiv (r+j)t \pmod{F_n} && \text{for some } F_{n-2}+1 \le r \le F_n \\ &\Leftrightarrow k \equiv (r-j)s \pmod{F_n} && \text{for some } 0 \le r \le F_{n-1} - 1 \\ &\Leftrightarrow k \equiv 1+(r-j)s \pmod{F_n} && \text{for some } F_{n-2} \le r \le F_n - 1 \\ &\Leftrightarrow k \equiv 1+(r+j)t \pmod{F_n} && \text{for some } 1 \le r \le F_{n-1}. \end{aligned}$$

Proof: The case where $j = 0$ in (1) has been proved in [2] and the other results follow easily from (1).

Given $r_1, r_2, \ldots, r_{n-2}$, to generate the Fibonacci word $w = w_n^{r_1 r_2 \cdots r_{n-2}}$, we first compute $k = \sum_{i=1}^{n-2} F_{i+1} r_i + 1$ and j satisfying

$$j \equiv \begin{cases} kF_{n-1} \pmod{F_n} & (n \text{ odd}), \\ kF_{n-1} - 1 \pmod{F_n} & (n \text{ even}), \end{cases}$$

and $1 \le j \le F_n$. Then $w = T^{js}(q_n)$ (see [2]); thus, any one of the first four conditions in Theorem 2 gives precisely the positions of the letter "a" in w. Hence, w can be constructed easily.

Besides using congruences, other methods of locating the letters are discussed in [4]; for example, using Zeckendorf representations and the golden ratio.

3. SHIFT OPERATION

It has been shown in [2] that Q_n consists of F_n distinct elements and, for any $w \in Q_n$, w, $T(w)$, ..., $T^{F_n - 1}(w)$ is a list of all these elements. In this way, every n^{th}-order Fibonacci word is a generator of Q_n.

4. ADJACENT TRANSPOSITION AND MINIMUM SUM

Let q_n, $n = 3, 4, \ldots, s, t$ be as in section 2. For $w = c_1 c_2 \cdots c_m$ where c_j equals a or b, we designate by $S(w)$ the sum of the indices j for which $c_j = a$ and, for $1 \le k \le m$, we put

$$h_k(w) = d_1 d_2 \cdots d_m$$

where $d_k = c_{k+1}$, $d_{k+1} = c_k$, with subscripts modulo m, and $d_j = c_j$, otherwise.

Theorem 3: For $1 \le j \le F_n$, let $k_j \equiv jt \pmod{F_n}$ and $1 \le k_j \le F_n$. Then

$$h_{k_j}(T^{(j-1)s}(q_n)) = T^{js}(q_n),\ 1 \le j \le F_n.$$

Proof: By Theorem 2, the positions of the letter "a" in $T^{(j-1)s}(q_n)$, $h_{k_j}(T^{(j-1)s}(q_n))$, $T^{js}(q_n)$ are, respectively,

$$jt,\ (j+1)t, \ldots, (j+F_{n-2}-1)t, \tag{2}$$

$$jt+1,\ (j+1)t, \ldots, (j+F_{n-2}-1)t, \tag{3}$$

$$(j+1)t, \ldots, (j+F_{n-2}-1)t,\ (j+F_{n-2})t, \tag{4}$$

modulo F_n. Since $(j+F_{n-2})t \equiv jt+1 \pmod{F_n}$, it follows that $h_{k_j}(T^{(j-1)s}(q_n)) = T^{js}(q_n)$.

Corollary 1: Let $u^{(0)} = q_n$, $u^{(j)} = h_{k_j}(u^{(j-1)})$, $1 \le j \le F_n - 1$. Then the sequence $u^{(0)}, u^{(1)}, \ldots, u^{(F_n-1)}$ is precisely the sequence $q_n, T^s(q_n), \ldots, T^{(F_n-1)s}(q_n)$ and consists of all n^{th}-order Fibonacci words.

More generally, given a word $w \in Q_n$, let $0 \le j \le F_n - 1$ be such that

$$j \equiv S(w) - S(q_n) \equiv S(w) - F_{n-2}(F_{n-2}+1)t/2 \pmod{F_n}.$$

[The last congruence follows from (4).] Then $w = T^{js}(q_n)$, so the sequence

$$v^{(0)} = w,\ v^{(r)} = h_{k_{j+r}}(v^{(r-1)}),\ 1 \le r \le F_n - 1 \tag{5}$$

(with subscript $j+r$ modulo F_n) coincides with the sequence

$$T^{js}(q_n),\ T^{(j+1)s}(q_n), \ldots,\ T^{(j+F_n-1)s}(q_n)$$

and consists of all the n^{th}-order Fibonacci words. The importance of this method is that, in the sequence (5), any two successive Fibonacci words differ only by a pair of consecutive letters (the first and the last letter in a word are considered as consecutive letters). This gives a simple way of generating all the n^{th}-order Fibonacci words from any given n^{th}-order Fibonacci word.

For example, with $n = 6$ and $w = bababbab$, we have $j \equiv 3$, and the sequence $v^{(r)}$ in (5) is given as follows:

r	$j+r \pmod{F_n}$	k_{j+r}	$v^{(r)}$
0	3		*bab***ab***bab*
1	4	4	*babbab***ab**
2	5	7	*b***ab***babba*
3	6	2	*bbab***ab***ba*
4	7	5	**b***babbab***a**
5	8	8	*ab***ab***babb*
6	1	3	*abbab***ab***b*
7	2	6	*abbabbab*

When the "**ab**" in bold face in each word in the last column is replaced by "*ba*," the next word is obtained. Note also that, in view of Corollary 1, the same list of Fibonacci words can be obtained by shifting the letters in the Fibonacci word five places to the left in each step.

Corollary 2: $S(T^{js}(q_n)) - S(T^{(j-1)s}(q_n)) = 1,\ 1 \le j \le F_n - 1$.

Proof: If $1 \le j \le F_n - 1$, then $k_j \not\equiv F_n$; thus,

$$S(T^{js}(q_n)) = S(h_{k_j}(T^{(j-1)s}(q_n))) = S(T^{(j-1)s}(q_n)) + 1$$

according to (2) and (3).

We have seen in [3] that $T^{(F_n-1)s}(q_n) = R(q_n)$. Therefore, we obtain the following corollary.

Corollary 3: $S(q_n) = \min\{S(w) : w \in Q_n\}$; $S(R(q_n)) = \max\{S(w) : w \in Q_n\}$.

Finally, it is easy to see that $S(q_n)$ and $S(w_n^0)$ satisfy, respectively, the following recursive relations:

$$S(q_n) = \begin{cases} S(q_{n-1}) + S(q_{n-2}) + F_{n-4}F_{n-1}, & \text{if } n \text{ is even,} \\ S(q_{n-1}) + S(q_{n-2}) + F_{n-3}F_{n-2}, & \text{if } n \text{ is odd,} \end{cases}$$

$$= \begin{cases} S(q_{n-1}) + S(q_{n-2}) + F_{n-3}F_{n-2} - 1, & \text{if } n \text{ is even,} \\ S(q_{n-1}) + S(q_{n-2}) + F_{n-3}F_{n-2}, & \text{if } n \text{ is odd,} \end{cases}$$

$$S(w_n^0) = S(w_{n-1}^0) + S(w_{n-2}^0) + F_{n-4}F_{n-1},$$

$n \ge 5$, and $S(q_3) = S(q_4) = 1$, $S(w_3^0) = S(w_4^0) = 2$. Also, we have $S(q_n) \equiv F_{n-2}(F_{n-2}+1)t/2 \pmod{F_n}$ according to (4).

5. FIBONACCI WORD PATTERNS

The *Fibonacci word patterns* $F^0(a, b)$ and $F^1(a, b)$ are defined by

$$F^0(a, b) = w_1 w_2 w_3^0 w_4^0 \ldots w_n^0 \ldots,$$

$$F^1(a, b) = w_1 w_2 w_3^1 w_4^1 \ldots w_n^1 \ldots,$$

where $w_1 = a$, $w_2 = b$. $F^1(a, b)$ has been studied by Turner ([8], [9]), and $F^1(b, ab)$ is a golden sequence.

The following embedding theorem has been proved in [4]. The notation $u[p:q]$ means the subword $a_p a_{p+1} \ldots a_q$ of the infinite word $u = a_1 a_2 a_3 \ldots$ where each a_n, $n \ge 1$, is a letter.

Theorem 4 (Embedding Theorem):

(a) Let all the Fibonacci words be listed in the following order:

$$w_1, w_2, w_3^0, T(w_3^0), \ldots, w_n^0, T(w_n^0), \ldots, T^{F_n-1}(w_n^0), \ldots.$$

Then the j^{th} Fibonacci word in the above list is $T^i(w_n^0)$ where n is the largest positive integer such that $F_{n+1} \le j$ and $i = j - F_{n+1}$. This Fibonacci word is precisely $F^0(a, b)[j : j + F_n - 1]$.

(b) Let all the Fibonacci words be listed in the following order:

$$w_1, w_2, T(w_3^1), T^2(w_3^1), \dots, T(w_n^1), T^2(w_n^1), \dots, T^{F_n}(w_n^1), \dots.$$

Then the j^{th} Fibonacci word in the above list is $T^i(w_n^1)$ where n is the largest positive integer such that $F_{n+1} \le j$ and $i = j - F_{n+1} + 1$. This Fibonacci word is precisely $F^1(a,b)[j - F_n + 1 : j]$.

In other words, all the Fibonacci words are embedded in the Fibonacci word patterns $F^0(a,b)$ and $F^1(a,b)$ in the above sense.

6. GENERATION WITHOUT REPETITIONS

Besides those methods described in Sections 3-5, we shall develop two additional methods of generating all the n^{th}-order Fibonacci words without repetitions.

Let R be the set of all words in $X^* \setminus \{1\}$ that contain no consecutive letters "a." As before, the first and the last letter in a word are considered as consecutive letters. Clearly, each Q_n is a subset of R. For $w \in R$, let $h(w)$ be the word obtained from w by wrapping w around then replacing each ba in w by ab and then unwrapping it. For example,

$$h(\mathbf{ba}b\mathbf{ba}bb) = abbabbb,$$
$$h(\mathbf{a}bb\mathbf{ba}b\mathbf{b}) = bbabba.$$

Only the letters in bold face have to be replaced.

Lemma 1: $h(w) = T(w)$ for all $w \in R$.

Proof: Let $w \in R$. Write

$$w = a_1 a_2 \cdots a_n$$
$$h(w) = c_1 c_2 \cdots c_n.$$

From the definition of h, we have

$$c_i = \begin{cases} b, & \text{if } a_i a_{i+1} = bb, \\ a, & \text{if } a_i a_{i+1} = ba, \\ b, & \text{if } a_i a_{i+1} = ab, \end{cases}$$

$1 \le i \le n$, with subscripts modulo n. Hence, $c_i = a_{i+1}$, $1 \le i \le n$, with subscripts modulo n. Therefore, $h(w) = T(w)$.

Theorem 5: Let $w \in Q_n$. Then the sequence

$$u^{(0)} = w,\ u^{(j)} = h(u^{(j-1)}),\quad j = 1, 2, \dots, F_n - 1,$$

is precisely the sequence $w, T(w), \dots, T^{F_n - 1}(w)$ and consists of all the n^{th}-order Fibonacci words.

Next we turn to a result that is related to the operations f and g defined in Section 1.

Lemma 2: Let $w \in X^* \setminus \{1\}$. Then

(a) $bg(w) = f(w)b$.

(b) $f(T(w)) = \begin{cases} g(w), & \text{if } w \text{ begins with an "}a\text{,"} \\ T(g(w)), & \text{if } w \text{ begins with a "}b\text{."} \end{cases}$

(c) $T(f(w)) = g(w)$.

Proof:

(a) We prove the result by induction on the length m of w. Clearly, the result holds for $m = 1$. Now assume that the result is true for some $m \geq 1$. Let $w \in X^* \setminus \{1\}$ have length m. Then

$$bg(aw) = bbg(w) = bf(w)b = f(aw)b,$$
$$bg(bw) = babg(w) = baf(w)b = f(bw)b,$$

by the induction hypothesis.

(b) By part (a), we have, for any $u \in X^*$,

$$f(T(au)) = f(ua) = f(u)b = bg(u) = g(au),$$
$$f(T(bu)) = f(ub) = f(u)ba = bg(u)a = T(abg(u)) = T(g(bu)).$$

Therefore, (b) holds.

(c) Clearly, this holds for w having length 1. Assume that w has length ≥ 1. Then

$$T(f(aw)) = T(bf(w)) = f(w)b = bg(w) = g(aw),$$
$$T(f(bw)) = T(baf(w)) = af(w)b = abg(w) = g(bw),$$

by part (a). Therefore, (c) follows.

With this lemma, we now have a method of generating Q_{n+1}, without repetition, from Q_n by means of f and g.

Let $n \geq 3$. List the images of the sequence $w_n^0, T(w_n^0), \ldots, T^{F_n-1}(w_n^0)$ under f and g in the following order:

$$f(w_n^0), g(w_n^0), \ldots, f(T^i(w_n^0)), g(T^i(w_n^0)), \ldots, f(T^{F_n-1}(w_n^0)), g(T^{F_n-1}(w_n^0))$$

Then take away $g(T^i(w_n^0))$ from the list if $T^i(w_n^0)$ begins with an "a" because, in this case, $g(T^i(w_n^0)) = f(T^{i+1}(w_n^0))$ according to Lemma 2(b). Since there are F_{n-2} n^{th}-order Fibonacci words beginning with an "a" (see [2]), it follows that there are F_{n+1} words left in the list. Now, according to Lemma 2, we see that the resulting sequence coincides with the sequence

$$w_{n+1}^0, T(w_{n+1}^0), \ldots, T^{F_n-1}(w_{n+1}^0).$$

REFERENCES

1. P. G. Anderson. Private communication, 1992.
2. W. Chuan. "Fibonacci Words." *The Fibonacci Quarterly* **30.1** (1992):68-76.
3. W. Chuan. "Symmetric Fibonacci Words." *The Fibonacci Quarterly* **31.3** (1993):251-55.
4. W. Chuan. "Embedding Fibonacci Words into Fibonacci Word Patterns." In *Proceedings of the Fifth International Conference on Fibonacci Numbers and Their Applications*, pp. 113-22. Ed. G. E. Bergum, A. N. Philippou, & A. F. Horadam. Dordrecht: Kluwer, 1993.

5. W. Chuan. "Extraction Property of the Golden Sequence." Preprint, 1993.
6. R. J. Hendel & S. A. Monteferrante. "Hofstadter's Extraction Conjecture." To appear in *The Fibonacci Quarterly.*
7. D. R. Hofstadter. "Eta-Lore," p. 13. First presented at the Stanford Math Club, Stanford, California, 1963.
8. J. C. Turner. "Fibonacci Word Patterns and Binary Sequences." *The Fibonacci Quarterly* **26.3** (1988):233-46.
9. J. C. Turner. "The Alpha and the Omega of the Wythoff Pairs." *The Fibonacci Quarterly* **27.1** (1989):76-86.

AMS Classification Numbers: 68R15, 20M05

❖❖❖