FORMULAS FOR $a + a^2 2^p + a^3 3^p + \cdots + a^n n^p$

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1. INTRODUCTION

Let $S_{a,p}(n) = a + a^2 2^p + a^3 3^p + \cdots + a^n n^p$, with $n \in N, p \in N$, and $a \in R$ $(a \neq 0, a \neq 1)$, where N and R are, respectively, the sets of positive integers and real numbers.

In [2] N. Gauthier used a calculus-based method to evaluate $S_{a,p}(n)$. He wrote $S_{a,p}(n)$ as a^n times a polynomial of degree p in n plus a term which is n-independent. The coefficients are then determined recursively.

In this paper methods similar to those used in [1] are employed to derive various formulas for $S_{a,p}(n)$. Recurrence formulas in terms of powers of n and of $n+1$ are given. Explicit expressions for $S_{a,p}(n)$ in determinant form in terms of n and of $n+1$ are then derived from these formulas. These determinants are finally used to write $S_{a,p}(n)$ in terms of polynomials of degree p in n and in $n+1$.

2. FORMULAS IN TERMS OF POWERS OF $n+1$

2.1 A Recurrence Formula

Let $n \in N$. For $k \in N$ and $a \in R$ $(a \neq 0, a \neq 1)$, let

$$S_{a,k}(n) = a + a^2 2^k + a^3 3^k + \cdots + a^n n^k = \sum_{r=0}^{n} a^r r^k$$

and take

$$S_{a,0}(n) = 1 + a + a^2 + \cdots + a^n = \frac{a^{n+1}-1}{a-1}.$$

Then

$$\begin{aligned} a^{n+1}(n+1)^k &= S_{a,k}(n+1) - S_{a,k}(n) \\ &= \sum_{r=0}^{n} a^{r+1}(r+1)^k - S_{a,k}(n) \\ &= \sum_{r=0}^{n}\left(a^{r+1}\sum_{i=0}^{k}\binom{k}{i} r^i\right) - S_{a,k}(n) \\ &= a\sum_{i=0}^{k}\left(\binom{k}{i}\sum_{r=0}^{n} a^r r^i\right) - S_{a,k}(n) \\ &= a\sum_{i=0}^{k}\binom{k}{i} S_{a,i}(n) - S_{a,k}(n). \end{aligned} \tag{2.1.1}$$

The equation

$$a(S+1)^k - S^k = a^{n+1}(n+1)^k, \tag{2.1.2}$$

in which the binomial power on the left-hand side is expanded and S^i $(i = 0, 1, 2, \ldots, k)$ are then replaced by $S_{a,i}(n)$, provides a mnemonic for (2.1.1).

For example, for $k = 1$, formula (2.1.2) gives

$$a(S+1) - S = a^{n+1}(n+1),$$

and so

$$(a-1)S_{a,1}(n) + a\left(\frac{a^{n+1}-1}{a-1}\right) = a^{n+1}(n+1).$$

Hence,

$$S_{a,1}(n) = \frac{a^{n+1}}{a-1}(n+1) - \frac{a}{(a-1)^2}(a^{n+1}-1). \tag{2.1.3}$$

Also, by (2.1.2), with $k = 2$,

$$a^{n+1}(n+1)^2 = a(S+1)^2 - S^2 = a(S^2+2S+1) - S^2,$$

which implies that

$$(a-1)S_{a,2}(n) + 2aS_{a,1}(n) + a\left(\frac{a^{n+1}-1}{a-1}\right) = a^{n+1}(n+1)^2.$$

Thus, by (2.1.3),

$$S_{a,2}(n) = \frac{a^{n+1}}{a-1}(n+1)^2 - \frac{2a^{n+2}}{(a-1)^2}(n+1) + \frac{a^{n+1}-1}{(a-1)^3}(a^2+a).$$

2.2 $S_{a,p}(n)$ as a Determinant

Let $p \in N$ and let $k = 1, 2, \ldots, p$ in (2.1.1). It follows, applying Cramer's rule to these p equations together with the equation $S_{a,0}(n) = \frac{a^{n+1}-1}{a-1}$, that

$$S_{a,p}(n) = \frac{a^{n+p}}{(a-1)^p}\begin{vmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & a^{-n}\left(\frac{a^{n+1}-1}{a-1}\right) \\ 1 & \frac{a-1}{a} & 0 & 0 & \cdots & 0 & 0 & n+1 \\ 1 & \binom{2}{1} & \frac{a-1}{a} & 0 & \cdots & 0 & 0 & (n+1)^2 \\ \vdots & \vdots & \vdots & & & & & \vdots \\ 1 & \binom{p-1}{1} & \binom{p-1}{2} & & \cdots & \binom{p-1}{p-2} & \frac{a-1}{a} & (n+1)^{p-1} \\ 1 & \binom{p}{1} & \binom{p}{2} & & \cdots & \binom{p}{p-2} & \binom{p}{p-1} & (n+1)^p \end{vmatrix} \tag{2.2.1}$$

$$= \frac{a^{n+p}}{(a-1)^p}p!\begin{vmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & a^{-n}\left(\frac{a^{n+1}-1}{a-1}\right) \\ \frac{1}{1!} & \frac{a-1}{a} & 0 & 0 & \cdots & 0 & 0 & \frac{n+1}{1!} \\ \frac{1}{2!} & \frac{1}{1!} & \frac{a-1}{a} & 0 & \cdots & 0 & 0 & \frac{(n+1)^2}{2!} \\ \vdots & \vdots & \vdots & & & & & \vdots \\ \frac{1}{(p-1)!} & \frac{1}{(p-2)!} & \frac{1}{(p-3)!} & & \cdots & \frac{1}{1!} & \frac{a-1}{a} & \frac{(n+1)^{p-1}}{(p-1)!} \\ \frac{1}{p!} & \frac{1}{(p-1)!} & \frac{1}{(p-2)!} & & \cdots & \frac{1}{2!} & \frac{1}{1!} & \frac{(n+1)^p}{p!} \end{vmatrix}. \tag{2.2.2}$$

2.3 $S_{a,p}(n)$ in Terms of a Polynomial

By expanding the determinant (2.2.2) with respect to the last column,

$$S_{a,p}(n) = \sum_{r=0}^{p-1} \alpha_r (n+1)^{p-r} + \alpha_p a^{-(n+1)}(a^{n+1}-1), \tag{2.3.1}$$

with $\alpha_0 = \frac{a^{n+1}}{a-1}$ and, for $r = 1, 2, \ldots, p$,

$$\alpha_r = \binom{p}{r} \frac{a^{n+1}}{a-1} \frac{a^r}{(a-1)^r} r!(-1)^r \begin{vmatrix} \frac{1}{1!} & \frac{a-1}{a} & 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{2!} & \frac{1}{1!} & \frac{a-1}{a} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & & & & \\ \frac{1}{(r-1)!} & \frac{1}{(r-2)!} & & & \cdots & \frac{1}{1!} & \frac{a-1}{a} \\ \frac{1}{r!} & \frac{1}{(r-1)!} & & & \cdots & \frac{1}{2!} & \frac{1}{1!} \end{vmatrix}.$$

Now, let $f_0(a) = 1$ and, for $r = 1, 2, 3, \ldots$,

$$f_r(a) = \frac{a^r}{(a-1)^r} r!(-1)^r \begin{vmatrix} \frac{1}{1!} & \frac{a-1}{a} & 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{2!} & \frac{1}{1!} & \frac{a-1}{a} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & & & & \\ \frac{1}{(r-1)!} & \frac{1}{(r-2)!} & & & \cdots & \frac{1}{1!} & \frac{a-1}{a} \\ \frac{1}{r!} & \frac{1}{(r-1)!} & & & \cdots & \frac{1}{2!} & \frac{1}{1!} \end{vmatrix}. \tag{2.3.2}$$

Then, by (2.3.1),

$$S_{a,p}(n) = \frac{a^{n+1}}{a-1} \sum_{r=0}^{p-1} \binom{p}{r} f_r(a)(n+1)^{p-r} + f_p(a)\left(\frac{a^{n+1}-1}{a-1}\right). \tag{2.3.3}$$

The real numbers $f_r(a)$, $r = 1, 2, 3, \ldots$, can also be calculated recursively in the following way. Consider, for $r \in N$,

$$a^n f_r(a) = \frac{a^{n+r}}{(a-1)^r} r!(-1)^r \begin{vmatrix} \frac{1}{1!} & \frac{a-1}{a} & 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{2!} & \frac{1}{1!} & \frac{a-1}{a} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & & & & \\ \frac{1}{(r-1)!} & \frac{1}{(r-2)!} & & & \cdots & \frac{1}{1!} & \frac{a-1}{a} \\ \frac{1}{r!} & \frac{1}{(r-1)!} & & & \cdots & \frac{1}{2!} & \frac{1}{1!} \end{vmatrix}$$

$$= \frac{a^{n+r}}{(a-1)^r} r! \begin{vmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ \frac{1}{1!} & \frac{a-1}{a} & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{1}{2!} & \frac{1}{1!} & \frac{a-1}{a} & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & & & & & & \\ \frac{1}{(r-1)!} & \frac{1}{(r-2)!} & & & \cdots & \frac{1}{1!} & \frac{a-1}{a} & 0 \\ \frac{1}{r!} & \frac{1}{(r-1)!} & & & \cdots & \frac{1}{2!} & \frac{1}{1!} & 0 \end{vmatrix}.$$

Observe that the last determinant differs from that of $S_{a,r}(n)$, as obtained by setting $p = r$ in (2.2.2), only with respect to the last column. It follows [cf. (2.1.1)] that $f_0(a), f_1(a), f_2(a), \ldots$ satisfy the recurrence formula

$$f_0(a) = 1,\ a\sum_{i=0}^{r}\binom{r}{i} f_i(a) - f_r(a) = 0 \ (r = 1, 2, 3, \ldots). \tag{2.3.4}$$

Here the equation

$$a(f+1)^r - f^r = 0, \tag{2.3.5}$$

in which the binomial power is expanded and f^r $(r = 0, 1, 2, 3, \ldots)$ are then replaced by $f_r(a)$, provides a mnemonic for (2.3.4).

Note that (2.3.4), with $a = 1$, is the well-known recurrence formula for the Bernoulli numbers. The real numbers $f_r(a)$, $r = 0, 1, 2, 3, \ldots$, could therefore be called the a-Bernoulli numbers. For example, by (2.3.2) or, recursively, by (2.3.5),

$$f_0(a) = 1,\ f_1(a) = \frac{-a}{a-1},\ f_2(a) = \frac{a + a^2}{(a-1)^2},\ \text{and } f_3(a) = \frac{-(a + 4a^2 + a^3)}{(a-1)^3}. \tag{2.3.6}$$

Hence, $1, -2, 6, -26$ are the first four 2-Bernoulli numbers.

3. FORMULAS IN TERMS OF POWERS OF n

Let $n \in N$. For $k \in N$ and $a \in R$ $(a \neq 0, a \neq 1)$, let

$$S_{a,k}(n) = a + a^2 2^k + a^3 3^k + \cdots + a^n n^k = \sum_{r=1}^{n} a^r r^k,\ S_{a,k}(0) = 0,$$

and take

$$S_{a,0}(n) = a + a^2 + \cdots + a^n = \frac{a^{n+1} - a}{a - 1}.$$

Then, arguing as in Section 2.1,

$$a^n n^k = S_{a,k}(n) - S_{a,k}(n-1) = S_{a,k}(n) - \frac{1}{a}\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i} S_{a,k}(n).$$

Hence,

$$a^{n+1} n^k = aS_{a,k}(n) - \sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i} S_{a,i}(n). \tag{3.1}$$

The equation

$$aS^k - (S-1)^k = a^{n+1} n^k,$$

in which the binomial power on the left-hand side is expanded and S^i $(i = 0, 1, 2, \ldots, k)$ are then replaced by $S_{a,i}(n)$, provides a mnemonic for (3.1).

Furthermore, methods similar to those employed in Sections 2.2 and 2.3 can be used to derive the following results from (3.1).

$$S_{a,p}(n)=\frac{a^{n+1}}{(a-1)^p}p!\begin{vmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & a^{-(n+1)}\left(\frac{a^{n+1}-a}{a-1}\right) \\ \frac{1}{1!} & a-1 & 0 & 0 & \cdots & 0 & 0 & \frac{n}{1!} \\ -\frac{1}{2!} & \frac{1}{1!} & a-1 & 0 & \cdots & 0 & 0 & \frac{n^2}{2!} \\ \frac{1}{3!} & -\frac{1}{2!} & \frac{1}{1!} & a-1 & \cdots & 0 & 0 & \frac{n^3}{3!} \\ \vdots & \vdots & & & & & & \vdots \\ \frac{(-1)^p}{(p-1)!} & \frac{(-1)^{p+1}}{(p-2)!} & & & \cdots & \frac{1}{1!} & a-1 & \frac{n^{p-1}}{(p-1)!} \\ \frac{(-1)^{p+1}}{p!} & \frac{(-1)^{p+2}}{(p-1)!} & & & \cdots & -\frac{1}{2!} & \frac{1}{1!} & \frac{n^p}{p!} \end{vmatrix},$$

and

$$S_{a,p}(n)=\frac{a^{n+1}}{a-1}\sum_{r=0}^{p-1}\binom{p}{r}g_r(a)n^{p-r}+g_p(a)\left(\frac{a^{n+1}-a}{a-1}\right), \tag{3.2}$$

with $g_0(a)=1$ and, for $r=1,2,3,\dots$,

$$g_r(a)=\frac{r!(-1)^r}{(a-1)^r}\begin{vmatrix} \frac{1}{1!} & a-1 & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{2!} & \frac{1}{1!} & a-1 & 0 & \cdots & 0 & 0 \\ \frac{1}{3!} & -\frac{1}{2!} & \frac{1}{1!} & a-1 & \cdots & 0 & 0 \\ \vdots & \vdots & & & & & \\ \frac{(-1)^r}{(r-1)!} & \frac{(-1)^{r+1}}{(r-2)!} & & & \cdots & \frac{1}{1!} & a-1 \\ \frac{(-1)^{r+1}}{r!} & \frac{(-1)^{r+2}}{(r-1)!} & & & \cdots & -\frac{1}{2!} & \frac{1}{1!} \end{vmatrix}.$$

The real numbers $g_r(a)$, $r=1,2,3,\dots$, can also be calculated recursively in a similar way as it is done in the case of $f_r(a)$, $r=1,2,3,\dots$, in Section 2.3. However, it is easier to observe that, by (2.3.3) and (3.2) (comparing n-free terms), $f_r(a)=ag_r(a)$ for each $r\in N$. Hence, by (3.2),

$$S_{a,p}(n)=\frac{a^{n+1}}{a-1}n^p+\frac{a^n}{a-1}\sum_{r=1}^{p-1}\binom{p}{r}f_r(a)n^{p-r}+f_p(a)\left(\frac{a^n-1}{a-1}\right), \text{ for } p>1. \tag{3.3}$$

For example, let $p=2$ in (3.3). Then, by (2.3.6),

$$\begin{aligned} S_{a,2}(n)&=\frac{a^{n+1}}{a-1}n^2+\frac{2a^n}{a-1}f_1(a)n+f_2(a)\left(\frac{a^n-1}{a-1}\right) \\ &=\frac{a^{n+1}}{a-1}n^2-\frac{2a^{n+1}}{(a-1)^2}n+\frac{(a+a^2)(a^n-1)}{(a-1)^3}. \end{aligned}$$

In particular,

$$S_{3,2}(n)=\sum_{r=1}^{n}3^r r^2=\frac{3^{n+1}}{2}n^2-\frac{3^{n+1}}{2}n+\frac{3^{n+1}}{2}-\frac{3}{2}.$$

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