GENERALIZATIONS OF SOME SIMPLE CONGRUENCES

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1. INTRODUCTION

Over many years in this journal there have appeared results concerning congruence and divisibility in relation to the Fibonacci and Lucas numbers. Here we take four such results and translate them to sequences which generalize the Fibonacci and Lucas sequences.

We hope that the nature of our results will demonstrate to the beginning Fibonacci enthusiast that there is scope to obtain further generalizations of a similar nature.

2. THE SEQUENCES

In the notation of Horadam [7] write

$$W_n = W_n(a, b; p, q), \tag{2.1}$$

meaning that

$$W_n = pW_{n-1} - qW_{n-2},\ W_0 = a, W_1 = b,\ n \geq 2. \tag{2.2}$$

We assume throughout that a, b, p, q are integers.

The auxiliary equation associated with (2.2) is

$$x^2 - px + q = 0, \tag{2.3}$$

whose roots

$$\alpha = \frac{p + \sqrt{p^2 - 4q}}{2}, \quad \beta = \frac{p - \sqrt{p^2 - 4q}}{2} \tag{2.4}$$

are assumed distinct. We write

$$\Delta = (\alpha - \beta)^2 = p^2 - 4q. \tag{2.5}$$

We shall be concerned with specializations of the following two sequences:

$$\begin{cases} U_n = W_n(0, 1; p, q), \\ V_n = W_n(2, p; p, q). \end{cases} \tag{2.6}$$

The sequences $\{U_n\}$ and $\{V_n\}$ are the fundamental and primordial sequences, respectively, generated by (2.2). They are natural generalizations of the Fibonacci and Lucas sequences and have been studied extensively, particularly by Lucas [11]. Further information can be found, for example, in [1], [7], and [10].

The Binet forms for U_n and V_n are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{2.7}$$

$$V_n = \alpha^n + \beta^n. \tag{2.8}$$

These sequences can be extended to negative subscripts by the use of the recurrence (2.2) or the Binet forms.

We will make use of the following well-known results which we state for easy reference:

$$q^n U_{-n} = -U_n, \tag{2.9}$$

$$q^n V_{-n} = V_n, \tag{2.10}$$

$$U_{2n} = U_n V_n, \tag{2.11}$$

$$\text{if } m|n \text{ then } U_m | U_n. \tag{2.12}$$

The following identities, which occur in Bergum and Hoggatt [1], will also be needed:

$$U_{n+k} + q^k U_{n-k} = U_n V_k, \tag{2.13}$$

$$U_{n+k} - q^k U_{n-k} = U_k V_n, \tag{2.14}$$

$$V_{n+k} + q^k V_{n-k} = V_n V_k, \tag{2.15}$$

$$V_{n+k} - q^k V_{n-k} = \Delta U_n U_k. \tag{2.16}$$

The sequences

$$\begin{cases} U_n = W_n(0, 1; p, -1), \\ V_n = W_n(2, p; p, -1), \end{cases} \tag{2.17}$$

are an important subclass of the sequences (2.6) and can be looked upon as an intermediate level of generalization of the Fibonacci and Lucas numbers in which $p = 1$. The specializations $p = 2$ and $p = 2x$ also yield cases of interest. For $p = 2$ see [4], [8], [15] and for $p = 2x$ see [9], [12], [13].

We use the U_n–V_n notation throughout to refer to the sequences (2.6) and to the sequences (2.17). There will be no ambiguity since we shall always indicate the set to which we are referring.

3. CONGRUENCE RESULT I

Singh [17] gives the following:

$$L_{2^n} \equiv 7 \pmod{40} \quad \text{for } n \geq 2. \tag{3.1}$$

Generalization: Let $\{U_n\}$ and $\{V_n\}$ be the sequences defined in (2.17). Then

$$V_{2^n} \equiv V_4 \pmod{\Delta U_6 U_{10}}, \quad n = 2, 4, 6, \ldots, \tag{3.2}$$

$$V_{2^n} \equiv V_8 \pmod{\Delta U_6 U_{20} V_2 (V_4 - 1)}, \quad n = 3, 5, 7, \ldots, \tag{3.3}$$

and so

$$V_{2^n} \equiv V_4 \pmod{\Delta U_2 U_6} \quad \text{for } n \geq 2. \tag{3.4}$$

Proof: We shall use the following, all of which can be proved using Binet forms:

$$V_{2^{k+1}} = V_{2^k}^2 - 2, \tag{3.5}$$

$$p(V_4 + 1) = U_6, \tag{3.6}$$

$$V_4 - 2 = \Delta p^2, \tag{3.7}$$

$$p(V_4^2 + V_4 - 1) = U_{10}, \tag{3.8}$$

$$p(V_8 + 1) = U_6(V_4 - 1), \tag{3.9}$$

$$\Delta p^2 V_2^2 = V_8 - 2, \tag{3.10}$$

$$pV_2(V_8^2 + V_8 - 1) = U_{20}. \tag{3.11}$$

Using (3.5) twice, we also obtain

$$V_{2^{k+2}} = V_{2^k}^4 - 4V_{2^k}^2 + 2. \tag{3.12}$$

Now (3.2) is true for $n = 2$ and if it is true for $n = k$ (even) then by (3.12) and the induction hypothesis

$$V_{2^{k+2}} = V_{2^k}^4 - 4V_{2^k}^2 + 2 \equiv V_4^4 - 4V_4^2 + 2 \pmod{\Delta U_6 U_{10}}.$$

But by (3.6)-(3.8),

$$(V_4^4 - 4V_4^2 + 2) - V_4 = (V_4 + 1)(V_4 - 2)(V_4^2 + V_4 - 1) = \Delta U_6 U_{10}.$$

This proves (3.2). Congruence (3.3) can be proved similarly by making use of (3.9)-(3.12).

From (2.16) we see that $V_8 - V_4 = \Delta U_2 U_6$, and (2.12) shows that $\Delta U_2 U_6$ divides both moduli in (3.2) and (3.3). This proves (3.4). □

Putting $V_n = L_n$ so that $U_n = F_n$, we see that (3.4) reduces to (3.1).

4. CONGRUENCE RESULT II

Berzsenyi [2] states that

$$F_{6n+1}^2 \equiv 1 \pmod{24}, \ n \text{ an integer}. \tag{4.1}$$

Generalization: Let $\{U_n\}$ and $\{V_n\}$ be the sequences defined in (2.17). Then

$$U_{6n+1}^2 \equiv 1 \pmod{U_4 U_6}, \ n \text{ an integer}. \tag{4.2}$$

Proof:

$$\begin{aligned} U_{6n+1}^2 - 1 &= (U_{6n+1} - U_1)(U_{6n+1} + U_1) \\ &= (U_{3n+1+3n} - U_{3n+1-3n})(U_{3n+1+3n} + U_{3n+1-3n}) \\ &= U_{3n} V_{3n} U_{3n+1} V_{3n+1}, \end{aligned} \tag{4.3}$$

where we have used (2.13) and (2.14) with $q = -1$.

Taking m to be an integer, we consider two cases:

Case 1. $n = 2m + 1$. Using (2.11), the right side of (4.3) becomes $U_{12m+6} U_{12m+8}$. Then by (2.12), $U_4 | U_{12m+8}$ and $U_6 | U_{12m+6}$ and (4.2) follows.

Case 2. $n = 2m$. Using (2.11), the right side of (4.3) becomes $U_{12m} U_{12m+2}$. Since $U_4 | U_{12m}$, $U_6 | U_{12m}$, and $(U_4, U_6) = U_2 | U_{12m+2}$, then $U_4 U_6 | U_{12m} U_{12m+2}$ and (4.2) follows.

This completes the proof of (4.2). □

5. CONGRUENCE RESULT III

Freitag [5] gives the following:

$$L_{2\mathrm{p}^k} \equiv 3 \pmod{10} \tag{5.1}$$

for all primes $\mathrm{p} \geq 5$ and natural numbers k. We caution against confusing the prime p with the parameter p.

Generalization: Let $\{U_n\}$ and $\{V_n\}$ be the sequences defined in (2.17). Then

$$V_{n+12} \equiv V_n \pmod{40}. \tag{5.2}$$

Proof: Using (2.16), we see that

$$V_{n+12} - V_n = (p^2+4)U_6U_{n+6} = p(p^2+1)(p^2+3)(p^2+4)U_{n+6}. \tag{5.3}$$

The right side of (5.3) is divisible by 40 since 5 divides either p, p^2+1, or p^2+4 and $8|p(p^2+4)$ if p is even while $8|(p^2+1)(p^2+3)$ if p is odd.

To see that (5.2) generalizes (5.1) we note that, as observed in Bruckman [3],

$$2\mathrm{p}^k \equiv 2 \text{ or } -2 \pmod{12} \tag{5.4}$$

for all primes $\mathrm{p} \geq 5$. Now, since $L_2 = L_{-2} = 3$, (5.1) follows from (5.2) and (5.4). □

6. A DIVISIBILITY RESULT

Grassi [6] gives the following:

$$12|(F_{4n-2} + F_{4n} + F_{4n+2}), \tag{6.1}$$

$$168|(F_{8n-4} + F_{8n} + F_{8n+4}). \tag{6.2}$$

Generalization: Let $\{U_n\}$ and $\{V_n\}$ be the sequences defined in (2.6). Then for $n \geq 0$, $k \geq 1$,

$$U_{2k-1}V_{4k-2}V_{6k-3}|(q^{4k-2}U_{(4k-2)(2n-1)} - q^{2k-1}U_{(8k-4)n} + U_{(4k-2)(2n+1)}), \tag{6.3}$$

$$V_{2k}V_{4k}U_{6k}|(q^{4k}U_{4k(2n-1)} + q^{2k}U_{8kn} + U_{4k(2n+1)}). \tag{6.4}$$

Proof: We prove (6.3) by using reasoning similar to Mana [14]. Fixing k and denoting the dividend by $G_n^{(k)}$ we have, by (2.9),

$$\begin{aligned} G_0^{(k)} &= q^{4k-2}U_{-(4k-2)} + U_{4k-2} \\ &= -U_{4k-2} + U_{4k-2} = 0. \end{aligned}$$

Also

$$\begin{aligned} G_1^{(k)} &= q^{4k-2}U_{4k-2} - q^{2k-1}U_{8k-4} + U_{12k-6} \\ &= q^{4k-2}U_{2k-1}V_{2k-1} + U_{2k-1}V_{10k-5} && \text{[by (2.11) and (2.14)]} \\ &= U_{2k-1}V_{6k-3}V_{4k-2} && \text{[by (2.15)].} \end{aligned}$$

Now $\{G_n^{(k)}\}$ can be regarded as the sum of three sequences each satisfying the same homogeneous linear second-order recurrence relation with integer coefficients (see Shannon and Horadam [16]).

Hence, $\{G_n^{(k)}\}$ also satisfies this second-order recurrence. Therefore, since $U_{2k-1}V_{4k-2}V_{6k-3}|G_0^{(k)}$ and $U_{2k-1}V_{4k-2}V_{6k-3}|G_1^{(k)}$, then $U_{2k-1}V_{4k-2}V_{6k-3}|G_n^{(k)}$ for all $n \geq 0$. Since k was arbitrary, the proof of (6.3) is complete. The proof of (6.4) is similar. □

Taking $\{U_n\} = \{F_n\}$, $\{V_n\} = \{L_n\}$ and putting $k = 1$, we see that (6.3) and (6.4) reduce to (6.1) and (6.2), respectively.

7. CONCLUDING COMMENTS

We have chosen an assortment of results requiring essentially different methods of proof. For the most part, the moduli or divisors in question are products of terms from the relevant sequences. We feel that with this observation there is scope for the beginner to discover generalizations of a similar nature.

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