ON THE GENERAL LINEAR RECURRENCE RELATION

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The general m^{th}-order linear recurrence relation can be written as

$$R_n = \sum_{i=1}^{m} a_i R_{n-i}, \quad \text{for } m \ge 2, \tag{1}$$

where the a_i's are any complex numbers, with $a_m \neq 0$. If suitable initial values $R_{-(m-2)}$, $R_{-(m-3)}$, $\ldots, R_0, R_1$ are specified, the sequence $\{R_n\}$ is uniquely determined for all integral n.

The auxiliary equation of (1) is

$$x^m = \sum_{i=1}^{m} a_i x^{m-i}. \tag{2}$$

Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be the m roots, assumed distinct, of (2) and define $\overline{\alpha}_j$ by

$$\overline{\alpha}_j = \prod_{\substack{i=1 \\ i \neq j}}^{m} (\alpha_j - \alpha_i).$$

Then the *fundamental* $\{U_n\}$ and *primordial* $\{V_n\}$ sequences that satisfy (1) are given by the following Binet formulas [1]. For any integer n, we have

$$U_n = \sum_{j=1}^{m} \frac{\alpha_j^{n+m-2}}{\overline{\alpha}_j} \quad \text{and} \quad V_n = \sum_{j=1}^{m} \alpha_j^n, \tag{3}$$

so that $U_{-(m-2)} = U_{-(m-3)} = \cdots = U_{-1} = U_0 = 0$ and $U_1 = 1$. Also $V_1 = a_1$ and

$$V_i = a_1 V_{i-1} + \cdots + a_{i-1} V_1 + i a_i, \quad \text{for } 1 \le i \le m. \tag{4}$$

In this paper we answer a question of Jarden, who in his book [2] (p. 88), see also [1], asked for the value of $U_{2n} - U_n V_n$ for the m^{th}-order linear recurrence relation. For example, when $m = 2$, where $a_1 = a_2 = 1$, $\{U_n\}$ and $\{V_n\}$ are the Fibonacci and Lucas sequences, respectively. In this case, we have

$$U_{2n} - U_n V_n = 0.$$

For the general third- and fourth-order linear recurrence relations we have, respectively,

$$U_{2n} - U_n V_n = a_3^n U_{-n} \quad \text{and} \quad U_{2n} - U_n V_n = (-1)^n a_4^n \{U_{-n} V_{-n} - U_{-2n}\}.$$

For the general m^{th}-order linear recurrence relation, we have the following, very appealing theorem.

Theorem: For any integer n, and $m \geq 2$, we have

$$U_{2n} - U_n V_n = (-1)^{(m+1)(n+1)} a_m^n \sum_{i=0}^{m-2} \sum_{\pi(i)} \frac{(-1)^k}{k_1!k_2!\ldots k_i!1^{k_1}2^{k_2}\ldots i^{k_i}} V_{-n}^{k_1} V_{-2n}^{k_2} \cdots V_{-in}^{k_i} U_{-(m-2-i)n},$$

where a_m is the constant term in the auxiliary equation and the inner summation is taken over all partitions of $i = 1k_1 + 2k_2 + \cdots + ik_i$ so that k_j is the number of parts of size j. Here, $k = k_1 + k_2 + \cdots + k_i$ is the total number of parts in the partition. The coefficient of $U_{-(m-2-i)n}$, inside the second summation sign, is taken to be 1 when $i = 0$.

In order to prove the above theorem, we use the following lemma.

Lemma: Using the above notation, we have

$$\sum_{\pi(i)} \frac{(-1)^k}{k_1!k_2!\ldots k_i!1^{k_1}2^{k_2}\ldots i^{k_i}} V_{-1}^{k_1} V_{-2}^{k_2} \ldots V_{-i}^{k_i} = \frac{a_{m-i}}{a_m} \quad \text{for } 0 \leq i \leq (m-1),$$
$$= -\frac{1}{a_m} \quad \text{for } i = m.$$

Proof of Lemma: First, we note that

$$\exp\left\{-\left(\frac{V_{-1}}{1}x + \frac{V_{-2}}{2}x^2 + \frac{V_{-3}}{3}x^3 + \cdots\right)\right\} = \sum_{i=0}^{\infty} x^i \sum_{\pi(i)} \frac{(-1)^k}{k_1!k_2!\ldots k_i!1^{k_1}2^{k_2}\ldots i^{k_i}} V_{-1}^{k_1} V_{-2}^{k_2} \ldots V_{-i}^{k_i}. \tag{5}$$

Therefore, we need to evaluate the function,

$$f(x) = \sum_{i=1}^{\infty} \frac{V_{-i}}{i} x^i.$$

Using the fact that $\{V_n\}$ satisfies the recurrence relation (1), with the help of (4) it is not hard to see that the generating function $g(x) = \sum_{n=0}^{\infty} V_{-n} x^n$, for V_{-n}, is given by

$$g(x) = \frac{ma_m + (m-1)a_{m-1}x + (m-2)a_{m-2}x^2 + \cdots + 2a_2 x^{m-2} + a_1 x^{m-1}}{a_m + a_{m-1}x + \cdots + a_1 x^{m-1} - x^m}. \tag{6}$$

Letting

$$h(x) = 1 + \frac{a_{m-1}}{a_m}x + \frac{a_{m-2}}{a_m}x^2 + \cdots + \frac{a_1}{a_m}x^{m-1} - \frac{1}{a_m}x^m, \tag{7}$$

from (6) and (7) we have

$$g(x) = m - \frac{h'(x)}{h(x)}x. \tag{8}$$

Now, since $V_0 = m$, from (8) we have

$$-\sum_{n=1}^{\infty} V_{-n} x^{n-1} = \frac{m - g(x)}{x} = \frac{h'(x)}{h(x)}.$$

Integrating, and using $h(0)=1$ to eliminate the constant of integration, we have

$$-\sum_{n=1}^{\infty}\frac{V_{-n}}{n}x^n = \log h(x).$$

Therefore,

$$\exp\left\{-\sum_{n=1}^{\infty}\frac{V_{-n}}{n}x^n\right\} = h(x). \tag{9}$$

So, from (5) and (9) we have

$$h(x)=\sum_{i=0}^{\infty}x^i\sum\frac{(-1)^k}{k_1!k_2!\dots k_i!1^{k_1}2^{k_2}\dots i^{k_i}}V_{-1}^{k_1}V_{-2}^{k_2}\dots V_{-i}^{k_i}. \tag{10}$$

Using the expression for $h(x)$ given by (7), we can equate the coefficients of x in (10) to complete the proof of the lemma. □

Proof of Theorem: From the Binet formulas (3) for U_n and V_n, we have

$$\begin{aligned}U_{2n}-U_nV_n &= \left(\frac{\alpha_1^{2n+m-2}}{\overline{\alpha}_1}+\frac{\alpha_2^{2n+m-2}}{\overline{\alpha}_2}+\dots+\frac{\alpha_m^{2n+m-2}}{\overline{\alpha}_m}\right)\\ &\quad -\left(\frac{\alpha_1^{n+m-2}}{\overline{\alpha}_1}+\frac{\alpha_2^{n+m-2}}{\overline{\alpha}_2}+\dots+\frac{\alpha_m^{n+m-2}}{\overline{\alpha}_m}\right)(\alpha_1^n+\alpha_2^n+\dots+\alpha_m^n)\\ &= -\sum_{i\neq j}\frac{\alpha_j^{n+m-2}\alpha_i^n}{\overline{\alpha}_j},\end{aligned} \tag{11}$$

where the summation is taken over all $1\le i, j\le m$, such that $i\neq j$. Therefore, to prove the theorem, we need to show that the right-hand side of the theorem is given by the right-hand side of (11). First, we require some new notation. The a_i in (2) are given by

$$a_i=(-1)^{i+1}\sum\alpha_1\alpha_2\dots\alpha_i,$$

where α_i are the roots of (2) and the summation is taken over all possible distinct products of i distinct α_j's. Now define $a_i(n)$ and $c_i(n)$ by

$$a_i(n)=(-1)^{i+1}\sum\alpha_1^n\alpha_2^n\dots\alpha_i^n \quad\text{and}\quad c_i(n)=\sum\alpha_1^n\alpha_2^n\dots\alpha_i^n,$$

so that $a_i(n)=(-1)^{i+1}c_i(n)$. Now, by the lemma, for any integer n, we have

$$\begin{aligned}\sum_{\pi(i)}\frac{(-1)^k}{k_1!k_2!\dots k_i!1^{k_1}2^{k_2}\dots i^{k_i}}V_{-n}^{k_1}V_{-2n}^{k_2}\dots V_{-in}^{k_i} &= \frac{a_{m-i}(n)}{a_m(n)} \quad \text{for } 0\le i\le (m-1),\\ &= -\frac{1}{a_m(n)} \quad \text{for } i=m.\end{aligned} \tag{12}$$

Using (12), we can rewrite the theorem as

$$U_{2n}-U_nV_n=(-1)^{(m+1)(n+1)}a_m^n\sum_{i=0}^{m-2}\frac{a_{m-i}(n)}{a_m(n)}U_{-(m-2-i)n}. \tag{13}$$

Since

$$a_m^n = (-1)^{(m+1)n} c_m(n), \qquad a_{m-i}(n) = (-1)^{m+i+1} c_{m-i}(n), \tag{14}$$

and

$$a_m(n) = (-1)^{m+1} c_m(n),$$

we have, from (13) and (14),

$$U_{2n} - U_n V_n = (-1)^{m+1} \sum_{i=0}^{m-2} (-1)^i c_{m-i}(n) U_{-(m-2-i)n}. \tag{15}$$

By the Binet formula,

$$U_{-(m-2-i)n} = \sum_{j=1}^{m} \frac{\alpha_j^{in-mn+2n+m-2}}{\overline{\alpha}_j},$$

which, when inserted into (15), gives

$$\begin{aligned} U_{2n} - U_n V_n &= (-1)^{m+1} \sum_{i=0}^{m-2} (-1)^i c_{m-i}(n) \sum_{j=1}^{m} \frac{\alpha_j^{in-mn+2n+m-2}}{\overline{\alpha}_j} \\ &= (-1)^{m+1} \sum_{j=1}^{m} \frac{\alpha_j^{2n+m-2}}{\overline{\alpha}_j} \sum_{i=0}^{m-2} (-1)^i c_{m-i}(n) \alpha_j^{(i-m)n}. \end{aligned} \tag{16}$$

Now we note that

$$\begin{aligned} \left(x + \frac{1}{\alpha_1^n}\right)\left(x + \frac{1}{\alpha_2^n}\right) \cdots \left(x + \frac{1}{\alpha_m^n}\right) &= \sum_{i=0}^{m} \frac{c_i(n)}{c_m(n)} x^i \\ &= \sum_{i=0}^{m} \frac{c_{m-i}(n)}{c_m(n)} x^{m-i} \end{aligned} \tag{17}$$

So if we let $x = -1/\alpha_j^n$ in (17), for any $j = 1, 2, \ldots, m$, we have

$$\sum_{i=0}^{m} (-1)^i c_{m-i}(n) \alpha_j^{(i-m)n} = 0. \tag{18}$$

From (18), we easily obtain

$$(-1)^{m+1} \sum_{i=0}^{m-2} (-1)^i c_{m-i}(n) \alpha_j^{(i-m)n} = -c_1(n) \alpha_j^{-n} + c_0(n). \tag{19}$$

Now we note that $c_0(n) = 1$ and $c_1(n) = \sum_{i=1}^m \alpha_i^n$. Therefore, using (19) in (16), we have

$$U_{2n} - U_n V_n = \sum_{j=1}^{m} \frac{\alpha_j^{2n+m-2}}{\overline{\alpha}_j} \left\{ -\sum_{i=1}^{m} \alpha_i^n \alpha_j^{-n} + 1 \right\} = -\sum_{j=1}^{m} \sum_{i=1}^{m} \frac{\alpha_j^{n+m-2} \alpha_i^n}{\overline{\alpha}_j} + \sum_{j=1}^{m} \frac{\alpha_j^{2n+m-2}}{\overline{\alpha}_j} = -\sum_{i \neq j} \frac{\alpha_j^{n+m-2} \alpha_i^n}{\overline{\alpha}_j}.$$

Which agrees with the right-hand side of (11). Hence, the theorem is proved. □

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1. A. G. Shannon. "Some Properties of a Fundamental Sequence of Arbitrary Order." *The Fibonacci Quarterly* **12.4** (1974):327-35.
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