NONZERO ZEROS OF THE HERMITE POLYNOMIALS ARE IRRATIONAL

P. R. Subramanian
Department of Nuclear Physics, University of Madras, Madras 600 025, India
(*Submitted July 1993*)

1. INTRODUCTION

The Hermite polynomials belong to the system of classical orthogonal polynomials [7, 10] and they are defined by means of the following relation [1, 7]:

$$\exp(2st - t^2) = \sum_{L=0}^{\infty} H_L(s)t^L / L!. \tag{1}$$

All the zeros of the Hermite polynomials are real, distinct, and are located in the open interval $(-\infty, \infty)$ [7, 10]. The Hermite polynomials of even degree have no zero at the origin, while each Hermite polynomial of odd degree has a simple zero at the origin [7]. The purpose of this article is to present an *elementary* proof that *all* the *nonzero* zeros of the Hermite polynomials are necessarily *irrational.*

2. BASIS OF THE PROOF

Our proof of the irrationality of the nonzero zeros of the Hermite polynomial $H_n(s)$ is based on the following facts:

(a) All the coefficients of the Hermite polynomials [1] are integers.

(b) A factor $2^m s^r$ can be pulled out of $H_n(s)$, $n \geq 1$, such that the remaining factor $R_{2k}^{(r)}(s)$ is an even polynomial in s of degree $2k$, still containing only integers as coefficients. The non-negative integers m, r, and k are given by

$$m = [(n+1)/2], \quad r = n - 2[n/2], \quad k = [n/2], \tag{2}$$

where $[t]$ is the greatest integer $\leq t$. Note that $m = r + k$, and that r is zero (unity) when n is even (odd).

(c) The constant term of $R_{2k}^{(r)}(s)$ is $(-1)^k(2k+2r-1)!!$, where

$$(2n-1)!! = 1 \cdot 3 \cdot 5 \cdots (2n-1), \quad n \geq 1. \tag{3}$$

We follow the convention that $(-1)!! = 1$. *All* the factors of the constant term of $R_{2k}^{(r)}(s)$ are *odd*.

(d) The leading coefficient (i.e., the coefficient of the highest power of s) in $R_{2k}^{(r)}(s)$ is 2^k, whose factors are of the form 2^c, $0 \leq c \leq k$, c being a nonnegative integer.

(e) The constant term of $R_{2k}^{(r)}(s)$ is *odd*, while all the other coefficients are *even* and nonzero.

(f) The zeros of $R_{2k}^{(r)}(s)$ are just the zeros of $H_n(s)$, $n \geq 1$, when n is even. If n is odd, the nontrivial zeros of $H_n(s)$ are simply the zeros of $R_{2k}^{(r)}(s)$, the trivial zero being the one located at the origin. The last result follows from the fact that $H_L(s)$ has a definite parity [10],

$$H_L(-s) = (-1)^L H_L(s), \ L \geq 0, \tag{4}$$

so that $H_{2M+1}(s)$, $M \geq 0$, is an odd polynomial in s and, hence, is zero at the origin.

(g) For all $p, q \in Z$, $2p \pm (2q+1) = 2(p \pm q) \pm 1 \neq 0$.

Thus (see [6], p. 81), for example,

$$\begin{aligned} H_6(s) &= 64s^6 - 480s^4 + 720s^2 - 120 \\ &= 8(8s^6 - 60s^4 + 90s^2 - 15), \end{aligned} \tag{5}$$

and

$$\begin{aligned} H_9(s) &= 512s^9 - 9216s^7 + 48384s^5 - 80640s^3 + 30240s \\ &= 32s(16s^8 - 288s^6 + 1512s^4 - 2520s^2 + 945). \end{aligned} \tag{6}$$

Using the power series expansion of the Hermite polynomials [1],

$$H_L(s) = \sum_{Q=0}^{[L/2]} \frac{(-1)^Q L!(2s)^{L-2Q}}{Q!(L-2Q)!}, \ L \geq 0, \tag{7}$$

and the relation

$$(2M)! = 2^M M!(2M-1)!!, \ M = 0, 1, 2, 3, \ldots, \tag{8}$$

we can prove the results (a)-(e) given above. In Section 5, we prove these results for the case in which n is even. Using similar arguments, one can easily establish the results for odd values of n also.

3. ZEROS CANNOT BE NONZERO INTEGERS

An immediate and interesting consequence of the result (e) of Section 2 is the fact that *no* nonzero integer (positive or negative) can be a zero of $H_n(s)$, since

$$R_{2k}^{(r)}(\text{integer}) = \text{even } \# \pm \text{odd } \# = \text{odd } \# \neq 0$$

(see result (g) of Section 2). Thus, $H_n(s)$, $n \geq 1$, is nonzero whenever s is an integer $\neq 0$. Hence, the zeros of $H_n(s)$ are neither positive nor negative integers.

4. NO RATIONAL ZEROS

If B/D, where B and D are integers, is a rational zero of a polynomial whose coefficients are all integers, and if B/D is in its lowest terms, then B must be a factor of the constant term and D must be a factor of the leading coefficient (see [5], Theorem 9-14, p. 303). Thus, a nontrivial rational zero of the Hermite polynomial, being a rational zero of $R_{2k}^{(r)}(s)$, whose coefficients are all integers, would be of the form B/D, where B = odd # and $D = 2^c$, where $c = 0, 1, 2, \ldots, k$. (Remember results (c) and (d) of Section 2.) The case $c = 0$ corresponds to an integer as a possible zero and, hence, can be ruled out (see Section 3). Using (7), it can be shown that

$$2^{n(c-1)} H_n(\text{odd } \#/2^c) = \text{odd } \# \neq 0, \ n \geq 1, \ c \geq 1. \tag{9}$$

For a proof, see Section 6. Since $2^{n(c-1)} \neq 0$, $s = \text{odd } \#/2^c$ cannot be a zero of $H_n(s)$. We conclude that the Hermite polynomial $H_n(s)$, $n \geq 1$, has *no* nonzero rational zeros.

5. PROOF OF CERTAIN RESULTS FROM SECTION 2

We now prove some of the statements given in Section 2 for the case in which n = even #. If the coefficient of s^{2N-2Q} in $H_{2N}(s)$, $N \geq 1$, is A_{2N-2Q}, then, from (7) with $L = 2N$,

$$A_{2N-2Q} = (-1)^Q (2N)! 2^{2N-2Q} / \{Q!(2N-2Q)!\}. \tag{10}$$

Now, when K is a positive integer, $(2K)! = (2K)(2K-1)(2K-2)(2K-3)\cdots 4\cdot 3\cdot 2\cdot 1$, and since $(-1)!! = 1$ (see result (c) of Section 2), we have

$$(2M)! = 2^M M!(2M-1)!!, \quad M = 0, 1, 2, 3, \ldots. \tag{8}$$

It follows from (8) and (10) that

$$A_{2N-2Q} = 2^N (-1)^Q \binom{N}{Q} 2^{N-Q} \{(2N-1)!!/(2N-2Q-1)!!\}. \tag{11}$$

In (11), the binomial coefficient $\binom{N}{Q}$ is necessarily a positive *integer*; the expression within the braces $\{\cdots\}$ is essentially an odd positive *integer*, since $Q \leq N$, and both Q and N are nonnegative integers. The phase factor $(-1)^Q$ is an odd integer $(= \pm 1)$. The factor 2^{N-Q} is *even* as long as $Q \neq N$; for the constant term of $H_{2N}(s)$, this quantity is just unity and, hence, odd (as $Q = N$). The leading coefficient of $H_{2N}(s)$ is $A_{2N} = 2^{2N}$ (since $Q = 0$). It is now clear from (11) that *all* the coefficients of $H_{2N}(s)$, $N \geq 1$, are *integers*. Moreover, a factor 2^N can be pulled out from *all* the coefficients of $H_{2N}(s)$, but still the coefficients of $R_{2N}^{(0)}(s)$ are *all* integers. The constant term of $R_{2N}^{(0)}(s)$ is *odd*, the leading coefficient is 2^N (= even #) and all the other coefficients are *even* numbers. Incidentally,

$$H_{2N}(0) = A_0 = (-1)^N 2^N (2N-1)!! \neq 0, \tag{12}$$

and, hence, $H_{2N}(s)$ can never be zero at the origin. It follows from (12) that the constant term of $R_{2N}^{(0)}(s)$ is just $(-1)^N (2N-1)!! = \text{odd } \# \neq 0$.

6. PROOF OF RELATION (9)

Let us now present the proof of (9) when n = even #. The proof is similar for the case when n = odd #.

Using (7) and (11), we have, with $c \geq 1$, $N \geq 1$,

$$\begin{aligned} 2^{2N(c-1)} H_{2N}(\text{odd } \#/2^c) = \sum_{Q=0}^{N} (-1)^Q \binom{N}{Q} 2^{Q(2c-1)} (\text{odd } \#)^{2N-2Q} \\ \times \{(2N-1)!!/(2N-2Q-1)!!\}. \end{aligned} \tag{13}$$

In the right-hand side of (13), the first term in the summation (i.e., $Q = 0$ term) is *odd*. For all the other terms, $Q \geq 1$, $Q(2c-1) \geq 1$, and, hence, $2^{Q(2c-1)} = \text{even } \# \geq 2$. Therefore, except for the first term, all the remaining terms are definitely *even*. Hence, $H_{2N}(\text{odd } \#/2^c) \neq 0$ and $s = \text{odd } \#/2^c$ cannot be a zero of $H_{2N}(s)$.

7. CONCLUSION

Zeros of the Hermite polynomials, if nonzero, are irrational. Using a computer program, we verified statements (b)-(e) of Section 2 and the results presented in Sections 3 and 4 for $n \le 12$. Readers familiar with Gaussian quadrature are practically aware that the nonzero zeros of $H_n(s)$, $2 \le n \le 20$ (say), are irrational [3, 4].

Recently, in connection with our work [9], we have been informed by the Editor of *The Journal of Number Theory* that the collected works of Professor I. Schur [8] contain a proof that the zeros of the Hermite and Laguerre polynomials are irrational. However, we are unable to access this material, independent of which our work was done, for verification. In fact, we learned about Schur's work [8] and Gow's work [2] only after [9] had already been accepted for publication and was subsequently rejected. The proof due to Professor Schur [8] *should be* delightful and, hopefully, *distinct* from ours!

ACKNOWLEDGMENTS

The author is grateful to Dr. G. Janhavi and Dr. M. Rajasekaran for fruitful discussions, and the referee for valuable comments and suggestions. We thank the UGC, New Delhi, for financial support through its COSIST and Special Assistance Programs. With all humility, this work is dedicated to Professor A. E. Labarre, Jr., for enlightening me through his beautiful book [5].

REFERENCES

1. W. W. Bell. *Special Functions for Scientists and Engineers*. London: Van Nostrand, 1968.
2. R. Gow. "Some Generalized Laguerre Polynomials Whose Galois Groups Are the Alternating Groups." *J. Number Theory* **31** (1989):201-07.
3. F. B. Hildebrand. *Introduction to Numerical Analysis*. 2nd ed. New Delhi: Tata McGraw-Hill, 1974.
4. Z. Kopal. *Numerical Analysis*, Appendix 4.3, pp. 530-33. New York: Wiley, 1955.
5. A. E. Labarre, Jr. *Elementary Mathematical Analysis*. Reading, Mass: Addison-Wesley, 1961.
6. L. Pauling & E. B. Wilson, Jr. *Introduction to Quantum Mechanics*. Tokyo: McGraw-Hill Kogakusha, 1935.
7. E. D. Rainville. *Special Functions*. New York: Macmillan, 1960.
8. I. Schur. *Collected Works*, Vol. 3, pp. 191-97, 227-33. Berlin: Springer, 1973.
9. P. R. Subramanian. "Zeros of the Simple Laguerre Polynomials of Degree Greater than Unity Are Irrational." Preprint.
10. G. Szegö. *Orthogonal Polynomials*. American Mathematical Society Colloquium Publications, 4th ed., Vol. 23. Providence, R.I.: American Mathematical Society, 1975.

AMS Classification Numbers: 33C45, 30C15

❖❖❖