ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-506 ***Proposed by Paul S. Bruckman, Highwood, IL***

Let

$$A=\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{5n+1}+\frac{1}{5n+4}\right) \quad \text{and} \quad B=\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{5n+2}+\frac{1}{5n+3}\right).$$

Evaluate A and B, showing that $A=\alpha B$.

H-507 ***Proposed by Mohammad K. Azarian, University of Evansville, Evansville, IN***

Prove that

$$\sum_{i=0}^{\infty}\sum_{j=0}^{n}\sum_{k=1}^{m}(-1)^i 2^{-(k+1)(i+j)}\left(\frac{n(i+1)(i+2)\cdots(i+n-1)}{j!(n-j)!}\right)(F_k)^{i+j}=1.$$

H-508 ***Proposed by H.-J. Seiffert, Berlin, Germany***

Define the Fibonacci polynomials by $F_0(x)=0$, $F_1(x)=1$, $F_n(x)=xF_{n-1}(x)+F_{n-2}(x)$, for $n\geq 2$. Show that, for all complex numbers x and y and all positive integers n,

$$F_n(x)F_n(y)=n\sum_{k=0}^{n-1}\frac{1}{k+1}\binom{n+k}{2k+1}(x+y)^k F_{k+1}\left(\frac{xy-4}{x+y}\right). \tag{1}$$

As special cases of (1), obtain the following identities:

$$F_n(x)F_n(x+1)=n\sum_{k=0}^{n-1}\frac{(-1)^{n-k+1}}{k+1}\binom{n+k}{2k+1}F_{k+1}(x^2+x+4); \tag{2}$$

$$F_n(x)F_n(4/x)=n\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\frac{1}{2k+1}\binom{n+2k}{4k+1}\left(\frac{x^2+4}{x}\right)^{2k}, \quad x\neq 0; \tag{3}$$

$$F_n(x)^2=n\sum_{k=0}^{n-1}\frac{(-1)^{n-k+1}}{k+1}\binom{n+k}{2k+1}(x^2+4)^k; \tag{4}$$

$$F_n(x)^2 = n\sum_{k=0}^{n-1} \frac{1}{k+1}\binom{n+k}{2k+1}\frac{x^{2k+2}-(-4)^{k+1}}{x^2+4}; \tag{5}$$

$$F_{2n-1}(x) = \sum_{k=0}^{2n-2} \frac{(-1)^k}{k+1}\binom{2n+k-1}{2k+1}x^k F_{k+1}(4/x). \tag{6}$$

SOLUTIONS

Get Hyper

H-491 ***Proposed by Paul S. Bruckman, Highwood, IL (Vol. 32, no. 5, November 1994)***

Prove the following identities:

$$F_{2n} = 2\binom{2n}{n}^{-1}\sum_{k=0}^{n-1}\binom{n-\frac{1}{2}}{k}\binom{n-\frac{1}{2}}{n-1-k}5^k, \quad n = 1, 2, \ldots; \tag{a}$$

$$F_{2n+1} = \binom{2n}{n}^{-1}\sum_{k=0}^{n}\binom{n-\frac{1}{2}}{k}\binom{n+\frac{1}{2}}{n-k}5^k, \quad n = 0, 1, 2, \ldots. \tag{b}$$

Solution by the proposer

Proof of Part (a): Let θ_n denote the sum given in the right member of the statement of part (a). It is more convenient to evaluate θ_{n+1}. Thus:

$$\theta_{n+1} = 2\binom{2n+2}{n+1}^{-1}\sum_{k=0}^{n}\binom{n+\frac{1}{2}}{k}\binom{n+\frac{1}{2}}{n-k}5^k, \quad n = 0, 1, \ldots. \tag{1}$$

Now

$$2\binom{2n+2}{n+1}^{-1} = 2(n+1)^2/(2n+2)(2n+1)\binom{2n}{n} = (n+1)/(2n+1)(-4)^n\binom{-\frac{1}{2}}{n} = \frac{n+1}{(-4)^n\binom{-\frac{3}{2}}{n}};$$

also,

$$\binom{n+\frac{1}{2}}{n-k}\binom{-\frac{3}{2}}{k} = (-1)^k\binom{n+\frac{1}{2}}{n-k}\binom{k+\frac{1}{2}}{k} = (-1)^k\binom{n+\frac{1}{2}}{n}\binom{n}{k} = (-1)^{n+k}\binom{-\frac{3}{2}}{n}\binom{n}{k}.$$

Therefore, after some simplification, θ_{n+1} is transformed to the following expression:

$$\theta_{n+1} = (n+1)\cdot 4^{-n}\sum_{k=0}^{n}\binom{n+\frac{1}{2}}{k}\binom{n}{k}\binom{-\frac{3}{2}}{k}^{-1}(-5)^k. \tag{2}$$

We recognize the last expression as a special case of the Hypergeometric Function. The Hypergeometric Function ${}_2F_1[{}^{a,b}_{c}; z]$ is defined if $c-a-b>0$ and $|z|<1$, as follows:

$${}_2F_1[{}^{a,b}_{c}; z] = \sum_{n\geq 0}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}. \tag{3}$$

It may happen that the series in (3) terminates after a finite number of terms, i.e., is a polynomial in z, in which case the restriction $|z|<1$ may be removed.

Since

$$\binom{n+\frac{1}{2}}{k}\binom{n}{k}\binom{-\frac{3}{2}}{k}^{-1}(-5)^k = \left(-n-\tfrac{1}{2}\right)_k (-n)_k \left(\tfrac{3}{2}\right)_k^{-1} \frac{5^k}{k!},$$

we see that

$$\theta_{n+1} = (n+1)\cdot 4^{-n}\cdot {}_2F_1\left[\begin{matrix} -n-\frac{1}{2}, -n \\ \frac{3}{2} \end{matrix}; 5\right]. \tag{4}$$

(Note that this expression is well-defined, since θ_{n+1} is a finite sum).

The following formula is given as Formula 15.1.10 in [1]:

$${}_2F_1\left[\begin{matrix} a, a+\frac{1}{2} \\ \frac{3}{2} \end{matrix}; z^2\right] = (2z)^{-1}(1-2a)^{-1}[(1+z)^{1-2a} - (1-z)^{1-2a}]. \tag{5}$$

Setting $a = -n-\frac{1}{2}$, $z = \sqrt{5}$ in (5), and comparing with (4) yields:

$$\begin{aligned}\theta_{n+1} &= (n+1)\cdot 4^{-n}(2\sqrt{5})^{-1}(2n+2)^{-1}[(1+\sqrt{5})^{2n+2} - (1-\sqrt{5})^{2n+2}] \\ &= 2^{2n+2}\cdot 4^{-n-1}\cdot 5^{-\frac{1}{2}}(\alpha^{2n+2} - \beta^{2n+2}) = F_{2n+2}.\end{aligned}$$

Then $\theta_n = F_{2n}$.

Proof of Part (b): Let Φ_n denote the sum given in the right member of the statement of part (b). Then, performing manipulations similar to those used in the proof of part (a), we obtain:

$$\Phi_n = (2n+1)\cdot 4^{-n}\sum_{k=0}^{n}\binom{n-\frac{1}{2}}{k}\binom{n}{k}\binom{-\frac{3}{2}}{k}^{-1}(-5)^k. \tag{6}$$

Then, as in the proof of part (a), Φ_n may be expressed in terms of the Hypergeometric Function as follows:

$$\Phi_n = (2n+1)\cdot 4^{-n}\cdot {}_2F_1\left[\begin{matrix} -n, -n+\frac{1}{2} \\ \frac{3}{2} \end{matrix}; 5\right]. \tag{7}$$

This time, we set $a = -n$, $z = \sqrt{5}$ in (5), which yields:

$$\begin{aligned}\Phi_n &= (2n+1)\cdot 4^{-n}(2\sqrt{5})^{-1}(2n+1)^{-1}[(1+\sqrt{5})^{2n+1} - (1-\sqrt{5})^{2n+1}] \\ &= 4^{-n}(2\sqrt{5})^{-1}2^{2n+1}(\alpha^{2n+1} - \beta^{2n+1}) = F_{2n+1}. \quad \text{Q.E.D.}\end{aligned}$$

Reference

1. *Handbook of Mathematical Functions*, ed. M. Abramowitz & I. A. Stegun, National Bureau of Standards, Ninth Printing, November, 1970.

Also solved by N. Jensen and H.-J. Seiffert.

More Sums

H-492 ***Proposed by H.-J. Seiffert, Berlin, Germany (Vol. 32, no. 5, November 1994)***

Define the Fibonacci polynomials by $F_0(x) = 0$, $F_1(x) = 1$, $F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$, for $n \geq 2$. Show that, for all complex numbers x and y and for all nonnegative integers n,

$$\sum_{k=0}^{[n/2]}\binom{n}{k}F_{n-2k}(x)F_{n-2k}(y) = z^{n-1}F_n(xy/z), \tag{1}$$

where $z=(x^2+y^2+4)^{1/2}$. [ ] denotes the greatest integer function.

As special cases of (1), obtain the following identities:

$$\sum_{k=0}^{[n/2]}\binom{n}{k}F_{n-2k}^2=(3^n-(-2)^n)/5, \tag{2}$$

$$\sum_{k=0}^{n}\binom{2n+1}{n-k}F_{2k+1}=5^n, \tag{3}$$

$$\sum_{k=0}^{n}\binom{2n}{n-k}F_{2k}F_{4k}=5^{n-1}(4^n-1) \tag{4}$$

$$\sum_{k=0}^{n}\binom{2n+1}{n-k}F_{2k+1}F_{4k+2}=5^n(2^{2n+1}+1), \tag{5}$$

$$\sum_{k=0}^{[n/2]}(-1)^k\binom{n}{k}F_{2n-4k}P_{n-2k}=F_n(6), \tag{6}$$

where $P_j=F_j(2)$ is the j^{th} Pell number,

$$\sum_{\substack{k=0\\(5,\,n-2k)=1}}^{[n/2]}(-1)^{[(n-2k+2)/5]}\binom{n}{k}=F_n. \tag{7}$$

The latter equation is the one given in H-444.

Solution by Norbert Jensen, Kiel, Germany

0. Note that the term on the right side of equation (1) is not defined for $z=0$, which can occur if $x, y\in\mathbb{C}$. However, the singularity is removable. For instance, it is easy to prove by induction that, for each $n\in\mathbb{N}_0$, there is a polynomial function $g_n:\mathbb{C}\times\mathbb{C}\times\mathbb{C}\to\mathbb{C}$ such that $z^{n-1}F_n(xy/z)=g_n(x,y,z^2)$. [The start of the induction is trivial. Then

$$g_{n+2}=z^{n+1}F_{n+2}(xy/z)=z^{n+1}((xy/z)F_{n+1}(xy/z)+F_n(xy/z))=xyg_{n+1}(x,y,z^2)+z^2g_n(x,y,z^2).]$$

1. Let $x\in\mathbb{R}$. Define

$$D(x)=x^2+4,\ d(x)=\sqrt{D(x)},$$

$$\alpha(x)=\frac{1}{2}(x+d(x)),\ \beta(x)=\frac{1}{2}(x-d(x)).$$

The explicit formula for the polynomials $F_n(x)$ is

$$F_n(x)=\frac{\alpha(x)^n-\beta(x)^n}{d(x)}\quad\text{for all } n\in\mathbb{N}_0.$$

2. Identity (1) will be derived from the following: Let $a, b, c, d\in\mathbb{R}$ be such that $ab=cd$. We prove, for all $n\in\mathbb{N}_0$,

2.1 $$\sum_{k=0}^{\left[\frac{n}{2}\right]}\binom{n}{k}(ab)^k[a^{n-2k}+b^{n-2k}-c^{n-2k}-d^{n-2k}]=(a+b)^n-(c+d)^n.$$

Proof: ***(i)*** Let $\lambda \in \mathbb{R}$. For all $n \in \mathbb{N}_0$, we have

$$\sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{k} \lambda^k (1+\lambda^{n-2k}) = \begin{cases} (1+\lambda)^n, & \text{if } n \text{ is odd,} \\ (1+\lambda)^n - \binom{n}{n/2} \lambda^{n/2} & \text{if } n \text{ is even.} \end{cases}$$

Proof of (i): Let $s = [(n-1)/2]$.

Case 1. n is odd. Then $s = (n-1)/2$. We have

$$\sum_{k=s+1}^{n} \binom{n}{k} \lambda^k = \sum_{k=s+1}^{n} \binom{n}{n-k} \lambda^k = \sum_{k=0}^{s} \binom{n}{k} \lambda^{n-k}.$$

Hence, the binomial theorem implies

$$(1+\lambda)^n = \sum_{k=0}^{s} \binom{n}{k} \lambda^k + \sum_{k=0}^{s} \binom{n}{k} \lambda^{n-k} = \sum_{k=0}^{s} \binom{n}{k} \lambda^k (1+\lambda^{n-2k}).$$

Case 2. n is even. Then $s = (n/2) - 1$. The proof is similar. The expression on the left side contains just the first s and the last s of the $s+1$ terms of the binomial sum.

Hence, we have to subtract the term $\binom{n}{n/2}\lambda^{n/2}$ on the right side. Q.E.D.

(ii) Let $b \neq 0$. Substituting λ by a/b and homogenizing the expression, we obtain

$$\sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{k} (ab)^k (a^{n-2k} + b^{n-2k}) = \begin{cases} (a+b)^n, & \text{if } n \text{ is odd,} \\ (a+b)^n - \binom{n}{n/2} (ab)^{n/2}, & \text{if } n \text{ is even.} \end{cases}$$

It is easy to check that the above equation is true for $b = 0$ as well.

(iii) Substituting c for a and d for b in the above equation and subtracting the equation obtained in this way from the one above we get, for all $n \in \mathbb{N}_0$:

$$\sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{k} (ab)^k [a^{n-2k} + b^{n-2k} - c^{n-2k} - d^{n-2k}] = (a+b)^n - (c+d)^n.$$

If n is odd, then $[(n-1)/2] = [n/2]$; if n is even, then $[(n-1)/2] = n/2 - 1$. However, the term $a^{n-2k} + b^{n-2k} - c^{n-2k} - d^{n-2k} = 0$ for $k = n/2$, so we can replace $[(n-1)/2]$ by $[n/2]$ in both cases. Q.E.D.

3. We prove equation (1) for all $x, y \in \mathbb{R}$ and all $n \in \mathbb{N}_0$.

3.1 Let $x, y \in \mathbb{R}$, and let

$$\begin{aligned} a &= (x+d(x))(y+d(y)), \\ b &= (x-d(x))(y-d(y)), \\ c &= (x+d(x))(y-d(y)), \\ d &= (x-d(x))(y+d(y)). \end{aligned}$$

Note that $ab = (x^2 - D(x))(y^2 - D(y)) = cd = 16$.

3.2 We have $d(x)d(y) = zd(xy/z)$.

Proof: $D(x)D(y)=(xy)^2+4x^2+4y^2+16=(xy)^2+4z^2=z^2[(xy/z)^2+4]=z^2D(xy/z)$ and $d(x)d(y)=zd(xy/z)$. Q.E.D.

3.3 For each $k\in\{0,1,\ldots,[n/2]\}$, we have

$$a^{n-2k}+b^{n-2k}-c^{n-2k}-d^{n-2k}=4^{n-2k}d(x)d(y)F_{n-2k}(x)F_{n-2k}(y).$$

Proof:

$$\begin{aligned}&a^{n-2k}+b^{n-2k}-c^{n-2k}-d^{n-2k}\\&=(x+d(x))^{n-2k}(y+d(y))^{n-2k}+(x-d(x))^{n-2k}(y-d(y))^{n-2k}\\&\quad-(x+d(x))^{n-2k}(y-d(y))^{n-2k}-(x-d(x))^{n-2k}(y+d(y))^{n-2k})\\&=[(x+d(x))^{n-2k}-(x-d(x))^{n-2k}](y+d(y))^{n-2k}\\&\quad-[(x+d(x))^{n-2k}-(x-d(x))^{n-2k}](y-d(y))^{n-2k}\\&=[(x+d(x))^{n-2k}-(x-d(x))^{n-2k}][(y+d(y))^{n-2k}-(y-d(y))^{n-2k}]\\&=4^{n-2k}d(x)d(y)F_{n-2k}(x)F_{n-2k}(y).\quad \text{Q.E.D.}\end{aligned}$$

3.4 We have $a+b=4z\alpha(xy/z)$, $c+d=4z\beta(xy/z)$.

Proof:

$$\begin{aligned}a+b&=(x+d(x))(y+d(y))+(x-d(x))(x-d(y))\\&=2(xy+d(x)d(y))=2z(xy/z+d(xy/z))=4z\alpha(xy/z)\end{aligned}$$

and

$$c+d=2(xy-d(x)d(y))=2z(xy/z-d(xy/z))=4z\beta(xy/z).\quad \text{Q.E.D.}$$

3.5 As by 3.1, we have $ab=(x^2-D(x))(y^2-D(y))=cd=16$ and we can apply equation **2.1** for a, b, c, and d chosen as above. Using 3.3 and 3.4, we obtain

$$\sum_{k=0}^{[\frac{n}{2}]}\binom{n}{k}16^k4^{n-2k}zd(xy/z)F_{n-2k}(x)F_{n-2k}(y)=4^nz^n(\alpha(xy/z)^n-\beta(xy/z)^n).$$

Dividing by $zd(xy/z)$ gives

$$\sum_{k=0}^{[\frac{n}{2}]}\binom{n}{k}F_{n-2k}(x)F_{n-2k}(y)=z^{n-1}\frac{\alpha(xy/z)^n-\beta(xy/z)^n}{d(xy/z)}=z^{n-1}F_n(xy/z).$$

As the term on the left side is, by the recursion formula, a polynomial expression in the variables x, y and the right side is a polynomial in x, y, z^2 by 0, the equation is true for all $x,y\in\mathbb{C}$. Here z can be any of the at most two possible roots of x^2+y^2+4.

4. Proof of identities (2)-(7).

4.1 Proof of identity (2): Take $x=y=1$. From the recursion formula, we obtain

4.1.1 $F_j(1)=F_j$ for each $j\in\mathbb{N}_0$. Now calculation shows: $z=\sqrt{6}$, $d(xy/z)=d(1/\sqrt{6})=5/\sqrt{6}$, $\alpha(xy/z)=\alpha(1/\sqrt{6})=\sqrt{6}/2$, and $\beta(xy/z)=\beta(1/\sqrt{6})=-2/\sqrt{6}$. Hence, we obtain

4.1.2 $z^{n-1}F_n(xy/z)=(\sqrt{6})^{n-1}F_n(1/\sqrt{6})=(\sqrt{6})^{n-1}\dfrac{\left(\frac{\sqrt{6}}{2}\right)^n-\left(-\frac{2}{\sqrt{6}}\right)^n}{5/\sqrt{6}}=\dfrac{3^n-(-2)^n}{5}.$

Now substitute 4.4.1, and 4.1.2 in (1). Q.E.D.

4.2 Proof of identity (3): Take $x=1$, $y=0$. Let $n=2m+1$ (and, finally, substitute m by n). We have $F_j(0)=(1^j-(-1)^n)/2=1$ or 0, according to whether j is odd or even for each $j\in\mathbb{N}_0$. Thus,

$$\textbf{4.2.1}\quad \sum_{k=0}^{[\frac{n}{2}]}\binom{n}{k}F_{n-2k}(x)F_{n-2k}(y)=\sum_{k=0}^{m}\binom{2m+1}{k}F_{2(m-k)+1}(1)F_{2(m-k)+1}(0)=\sum_{k=0}^{m}\binom{2m+1}{m-k}F_{2k+1}.$$

Calculation shows: $z=\sqrt{5}$, $d(xy/z)=d(0)=2$, $\alpha(xy/z)=\alpha(0)=1$, and $\beta(xy/z)=\beta(0)=-1$. Hence, we obtain

$$\textbf{4.2.2}\quad z^{n-1}F_n(xy/z)=(\sqrt{5})^{n-1}F_n(0)=(\sqrt{5})^{2m}F_{2m+1}(0)=5^m.$$

Now substitute 4.2.1 and 4.2.2 in (1). Q.E.D.

4.3 Proof of identity (4): Let $x=1$, $y=\sqrt{5}$. Let $n=2m$. Since $d(y)=d(\sqrt{5})=\sqrt{(5+4)}=3$, $\alpha(y)=(\sqrt{5}+3)/2=\alpha^2$, and $\beta(y)=-\beta^2$, we have $F_{2j}(y)=(\alpha^{4j}-(-\beta^2)^{2j})/3=(\sqrt{5}/3)F_{4j}$ for each $j\in\mathbb{N}_0$. Hence,

$$\textbf{4.3.1}\quad \sum_{k=0}^{[\frac{n}{2}]}\binom{n}{k}F_{n-2k}(x)F_{n-2k}(y)=\sum_{k=0}^{m}\binom{2m}{k}F_{2(m-k)}(x)F_{2(m-k)}(y)$$
$$=\sum_{k=0}^{m}\binom{2m}{m-k}F_{2k}(x)F_{2k}(y)=\sum_{k=0}^{m}\binom{2m}{m-k}F_{2k}F_{4k}(\sqrt{5}/3).$$

Now calculation shows: $z=\sqrt{10}$, $d(xy/z)=d(1/\sqrt{2})=3/\sqrt{2}$, $\alpha(xy/z)=\alpha(1/\sqrt{2})=\sqrt{2}$, and $\beta(xy/z)=\beta(1/\sqrt{2})=-1/\sqrt{2}$. Thus,

$$\textbf{4.3.2}\quad z^{n-1}F_n(xy/z)=(\sqrt{10})^{2m-1}F_{2m}(1/\sqrt{2})=\frac{10^m}{\sqrt{10}}\frac{2^m-(1/2)^m}{3/\sqrt{2}}=\frac{5^m}{\sqrt{5}}\frac{4^m-1}{3}.$$

Now substitute 4.3.1 and 4.3.2 in (1). Multiplying the equation by$3/\sqrt{5}$ completes the proof of identity (4). Q.E.D.

4.4 Proof of identity (5): Take $x=1, y=\sqrt{5}$ and let $n=2m+1$. Then $F_{2j+1}(y)=(\alpha^{4j+2}+\beta^{4j+2})/3=L_{4j+2}/3$ for each $j\in\mathbb{N}_0$. Hence

$$\textbf{4.4.1}\quad \sum_{k=0}^{[\frac{n}{2}]}\binom{n}{k}F_{n-2k}(x)F_{n-2k}(y)=\sum_{k=0}^{m}\binom{2m+1}{k}F_{2(m-k)+1}L_{r(m-k)+2}/3$$
$$=\sum_{k=0}^{m}\binom{2m+1}{m-k}F_{2k+1}L_{4k+2}/3.$$

Also

$$\textbf{4.4.2}\quad z^{n-1}F_n(xy/z)=z^{2m}F_{2m+1}(xy/z)=10^m\frac{\sqrt{2}^{2m+1}-(-1/\sqrt{2})^{2m+1}}{3/\sqrt{2}}=5^m\frac{2^{2m+1}+1}{3}.$$

Now substitute 4.4.1 and 4.4.2 in (1). Multiplying all equations by 3 completes the proof of identity (5). Q.E.D.

4.5 Proof of identity (6): Take $x = 3i$, where $i = \sqrt{-1}$, $y = 2$. Then $d(x) = i\sqrt{5}$, $\alpha(x) = (3i + i\sqrt{5})/2 = i\alpha^2, \beta(x) = i\beta^2$, $F_j(x) = ((i\alpha^2)^j - (i\beta^2)^j)/i\sqrt{5} = i^{j-1}F_{2j}$, and $F_j(y) = P_j$, for each $j \in \mathbb{N}_0$. Then

$$\textbf{4.5.1} \quad \sum_{k=0}^{[\frac{n}{2}]} \binom{n}{k} F_{n-2k}(x)F_{n-2k}(y) = \sum_{k=0}^{[\frac{n}{2}]} \binom{n}{k} i^{n-2k-1} F_{2n-4k} P_{n-2k}.$$

Calculation shows: $z = i$, $d(xy/z) = d(6) = 2\sqrt{10}$, $\alpha(xy/z) = \alpha(6) = 3 + \sqrt{10}$, and $\beta(xy/z) = \beta(6) = 3 - \sqrt{10}$. Hence

$$\textbf{4.5.2} \quad z^{n-1}F_n(xy/z) = i^{n-1}F_n(6).$$

Now substitute 4.5.1 and 4.5.2 in (1). Dividing all equations by i^{n-1} completes the proof of identity (6). Q.E.D.

4.6 Proof of identity (7): Let $x = i\alpha$, $y = i\beta$, where again $i = \sqrt{-1}$. We apply the recursion formula for the sequences $(F_j(i\alpha))$ and $(F_j(i\beta))$. In this way, we calculate $F_0(i\alpha)$, $F_1(i\alpha)$, ..., $F_{20}(i\alpha)$ and $F_0(i\beta), F_1(i\beta), \ldots, F_{20}(i\beta)$ and realize that both sequences $(F_j(i\alpha))$ and $(F_j(i\beta))$ have period 20. Thus

$$F_j(i\alpha)F_j(i\beta) = \begin{cases} 0, & \text{if } j \equiv 0 \pmod 5, \\ 1, & \text{if } j \equiv 1, 2, 8, 9 \pmod{10}, \\ -1, & \text{if } j \equiv 3, 4, 6, 7 \pmod{10}. \end{cases} \quad \text{for each } j \in \mathbb{N}_0.$$

In particular, $F_j(i\alpha)F_j(i\beta) = (-1)^{[(j+2)/5]}$ if j and 5 are coprime. Hence,

$$\textbf{4.6.1} \quad \sum_{k=0}^{[\frac{n}{2}]} \binom{n}{k} F_{n-2k}(i\alpha)F_{n-2k}(i\beta) = \sum_{\substack{k=0 \\ (5,\, n-2k)=1}} (-1)^{[(n-2k+2)/5]} \binom{n}{k}.$$

Also, $z = \sqrt{(-\alpha^2 - \beta^2 + 4)} = 1$, $xy/z = xy = 1$, so

$$\textbf{4.6.2} \quad z^{n-1}F_n(xy/z) = F_n(1) = F_n.$$

Now substitute 4.6.1 and 4.6.2 in (1). Q.E.D.

Also solved by P. Bruckman and the proposer.