ON THE APPROXIMATION OF IRRATIONAL NUMBERS WITH RATIONALS RESTRICTED BY CONGRUENCE RELATIONS

C. Elsner
Institut für Mathematik, Technische Universität Hannover, Welfengarten 1, Hannover, Germany
(*Submitted May 1994*)

1. INTRODUCTION

It is a well-known theorem of A. Hurwitz that for any real irrational number ξ there are infinitely many integers u and $v > 0$ satisfying

$$\left|\xi - \frac{u}{v}\right| \le \frac{1}{\sqrt{5}v^2}.$$

Usually this theorem is proved by using continued fractions; see Theorem 193 in [2].

S. Hartman [3] has restricted the approximating numbers $\frac{u}{v}$ to those fractions, where u and v belong to fixed residue classes a and b with respect to some modulus s. He proved the following:

For any irrational number ξ, any $s \ge 1$, and integers a and b, there are infinitely many integers u and $v > 0$ satisfying

$$\left|\xi - \frac{u}{v}\right| < \frac{2s^2}{v^2} \tag{1}$$

and

$$u \equiv a \mod s, \quad v \equiv b \mod s.$$

The special case $a = b = 0$ shows that the exponent 2 of s^2 is best-possible. In what follows, we are interested in the case where a and b are not both divisible by s; and we allow the denominators v to be negative. Using these conditions, S. Uchiyama [10] has published the following result:

For any irrational number ξ, any $s > 1$, and integers a and b, there are infinitely many integers u and $v \ne 0$ satisfying

$$\left|\xi - \frac{u}{v}\right| < \frac{s^2}{4v^2} \tag{2}$$

and

$$u \equiv a \mod s, \quad v \equiv b \mod s,$$

provided that it is not simultaneously $a \equiv 0 \mod s$ and $b \equiv 0 \mod s$.

Years before, J. F. Koksma [4] had proved a slightly weaker theorem. From the case $s = 2$ and Theorem 3.2 in L. C. Eggan's paper [1], it is clear that the constant $\frac{1}{4}$ in (2) is best-possible. It is proved by Eggan that for any $\sigma > 0$ and for any choice of the three types *odd/odd*, *odd/even*, or *even/odd* of the fractions $\frac{u}{v}$ there is an irrational number ξ so that no fraction of the chosen type satisfies

$$\left|\xi - \frac{u}{v}\right| < \frac{1-\sigma}{v^2}.$$

The case $s=2$ has been studied by other authors, see, e.g., [5], [6], [8], and [9]. In this paper we prove a smaller bound for $|\xi-\frac{u}{v}|$, assuming $u \equiv v \bmod s$ for some prime s. Actually, the result is a bit stronger.

Theorem 1: Let $0<\varepsilon \leq 1$, and let p be a prime with

$$p>\left(\frac{2}{\varepsilon}\right)^2;$$

h denotes any integer that is not divisible by p. Then, for any real irrational number ξ, there are infinitely many integers u and $v>0$ satisfying

$$\left|\xi-\frac{u}{v}\right| \leq \frac{(1+\varepsilon)p^{3/2}}{\sqrt{5}v^2} \tag{3}$$

and

$$u \equiv hv \not\equiv 0 \mod p.$$

To prove Theorem 1, we will apply the methods of S. Hartman [3] and S. Uchiyama [10]. It will be convenient to use the same notations as in Uchiyama's paper, but this is done for another reason: there is a small gap in the proof of Uchiyama's result stated above in (2). In what follows, we are concerned with the same difficulty at this point, and we will fill the gap.

2. AUXILIARY RESULTS

Apart from Hartman's method, we need two lemmas.

Lemma 1: Let $0<\varepsilon \leq 1$, and let p be a prime, and let w_1 and w_2 be integers with

$$p>\left(\frac{2}{\varepsilon}\right)^2, \tag{4}$$

$$0<w_1<p, \quad 0<w_2<p. \tag{5}$$

Then there are integers b, g_1, and g_2 satisfying

$$|b|<p, \tag{6}$$

$$0<|g_1| \leq (1+\varepsilon)p^{1/2}, \quad 0<|g_2| \leq (1+\varepsilon)p^{1/2}, \tag{7}$$

and

$$bw_1 \equiv g_1 \mod p, \quad bw_2 \equiv g_2 \mod p. \tag{8}$$

Lemma 2: There are no integers x and y with $xy>0$ satisfying simultaneously the following conditions:

$$\frac{1}{xy} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{x^2}+\frac{1}{y^2}\right), \quad \frac{1}{x(x+y)} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{x^2}+\frac{1}{(x+y)^2}\right).$$

Proof of Lemma 1: We try to solve a linear system of equations

$$\left.\begin{aligned} w_1x-y_1+py_2&=0 \\ w_2x-y_3+py_4&=0 \end{aligned}\right\} \tag{9}$$

with integers x, y_1, y_2, y_3, and y_4, where

$$|x| < p,$$

$$0 < |y_1|, |y_3| \le (1+\varepsilon)p^{1/2},$$

and

$$0 \le |y_2|, |y_4| \le 2kp.$$

Hence, $b = x$, $g_1 = y_1$, and $g_2 = y_3$ satisfy (6), (7), and (8) in our lemma. But first we do need an auxiliary inequality:

From $p > \left(\frac{2}{\varepsilon}\right)^2$, we conclude that

$$\frac{1}{\sqrt{p}} + \frac{1}{p} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

For $0 \le t \le \frac{1}{2}$, we have $\frac{1}{\sqrt{1-t}} \le 1+t$; hence,

$$\frac{1}{\sqrt{p}} + \frac{1}{\sqrt{1-\frac{1}{p}}} < 1+\varepsilon \quad (p \ge 2).$$

This is equivalent to

$$p^2 < (p-1)\left(2\left(\frac{1+\varepsilon}{2}\sqrt{p}-1\right)+1\right)^2.$$

It follows that

$$p^2 < (p-1)\left(2\left[\frac{1+\varepsilon}{2}\sqrt{p}\right]+1\right)^2, \tag{10}$$

where $[\alpha]$ denotes the integral part of α for nonnegative real numbers α. The integers X_1, X_2, and X_3 are given by

$$X_1 = \frac{p-1}{2}, \quad X_2 = \left[\frac{1+\varepsilon}{2}p^{1/2}\right], \quad X_3 = kp, \tag{11}$$

where k is a sufficiently large positive integer satisfying

$$\frac{\left(p(p-1)+2\left[\frac{1+\varepsilon}{2}p^{1/2}\right]+2kp^2+1\right)^2}{(2kp+1)^2} \le p^2\frac{p}{p-1}.$$

By (10) and (11), this implies

$$\frac{(2pX_1+2X_2+2pX_3+1)^2}{(2X_3+1)^2} < p(2X_2+1)^2$$

or

$$(2(pX_1+X_2+pX_3)+1)^2 < (2X_1+1)(2X_2+1)^2(2X_3+1)^2. \tag{12}$$

There are $(2X_1+1)(2X_2+1)^2(2X_3+1)^2$ different sets of integers x, y_1, y_2, y_3, and y_4 with

$$-X_1 \le x \le X_1, \quad -X_2 \le y_1, y_3 \le X_2, \quad -X_3 \le y_2, y_4 \le X_3.$$

We denote the left-hand sides of the two forms in (9) by f_1 and f_2; for each such set of integers, we have

$$-(pX_1+X_2+pX_3)\le f_1, f_2 \le pX_1+X_2+pX_3.$$

Here we have applied (5). There are at most $(2(pX_1+X_2+pX_3)+1)^2$ different sets f_1 and f_2. But now, by (12) and the box principle, there must be two distinct sets of five numbers x, y_1, y_2, y_3, and y_4 that correspond to the same set f_1 and f_2; their difference gives a nontrivial solution of (9) where, by (11):

$$0\le |x| < p, \tag{13}$$

$$0\le |y_1|, |y_3| \le (1+\varepsilon)p^{1/2}, \tag{14}$$

and

$$0\le |y_2|, |y_4| \le 2kp.$$

To finish the proof, we must show that $y_1\ne 0$ and $y_3\ne 0$. We assume the contrary for y_1, which gives $w_1x\equiv 0 \bmod p$ from the first equation in (9). Since p is a prime and $0<w_1<p$ by (5), this holds if and only if $x\equiv 0 \bmod p$. This means, by (13), that $x=0$. Thus, the first equation in (9) becomes $py_2=0$ or $y_2=0$ and the second one becomes $-y_3+py_4=0$ or $y_3\equiv 0 \bmod p$. Now we apply the condition $\sqrt{p}>2$ from (4), which yields

$$|y_3| \overset{(14)}{\le} (1+\varepsilon)p^{1/2} \le 2p^{1/2} < p;$$

hence, $y_3=0$.

We have obtained $x=y_1=y_2=y_3=y_4=0$, which contradicts our construction of a nontrivial solution of (9). We proceed in the same way if we assume $y_3=0$. Thus, the proof of Lemma 1 is complete.

Proof of Lemma 2: (See "Hilfssatz 2.2", Ch. 10, in [7].) Without loss of generality, we may assume $x>0$ and $y>0$; hence, from the two inequalities stated in Lemma 2, we have

$$0\ge x^2+y^2-xy\sqrt{5} \quad\text{and}\quad 0\ge (2-\sqrt{5})(x^2+xy)+y^2.$$

The sum of these inequalities gives

$$0\ge 2\left(\frac{3-\sqrt{5}}{2}x^2+(1-\sqrt{5})xy+y^2\right)=2\left(\frac{\sqrt{5}-1}{2}x-y\right)^2.$$

It follows that $2y=(\sqrt{5}-1)x$, which is impossible for $x\ne 0$.

3. PROOF OF THEOREM 1

Any real irrational number ξ is represented by its continued fraction expansion, that is, $\xi=[a_0;a_1,a_2,\ldots]$, where $a_0\in\mathbb{Z}$, $a_n\in\mathbb{Z}_{>0}$ $(n\ge 1)$. p_n and q_n from the n^{th} convergent $\frac{p_n}{q_n}$ with

$$\left|\xi-\frac{p_n}{q_n}\right|\le\frac{1}{q_n^2} \tag{15}$$

satisfy the recurrences

$$\begin{array}{llll} p_{-1}=1, & p_0=a_0, & p_n=a_np_{n-1}+p_{n-2} & (n\geq 1), \\ q_{-1}=0, & q_0=1, & q_n=a_nq_{n-1}+q_{n-2} & (n\geq 1). \end{array}$$

It is well known that

$$p_{n-1}q_n-p_nq_{n-1}=(-1)^n \tag{16}$$

holds for all integers $n\geq 1$. According to the usual notations, we have to distinguish carefully the notation p_m (with index m) and p, which denotes the prime modulus in Theorem 1.

Now, following the idea of Hartman, we consider for $n\geq 1$ a small system of congruences

$$\left.\begin{array}{r} p_nx+p_{n-1}y\equiv a \mod s, \\ q_nx+q_{n-1}y\equiv b \mod s, \end{array}\right\} \tag{17}$$

where $s\geq 2$ is some positive integer and a and b are fixed integers such that there is not simultaneously $a\equiv 0 \mod s$ and $b\equiv 0 \mod s$. It is easily proved by (16) that a solution of (17) is given by $x=t_{n-1}$ and $y=t_n$, where the integers t_{n-1} and t_n are determined by

$$t_m\equiv(-1)^m(aq_m-bp_m) \mod s. \tag{18}$$

In what follows, we consider only the sequence of all *even* integers $n>0$. In (17) and (18), we put $s=p$ and $a=hb$ and so, for even integers n, we compute t_n and t_{n-1} by

$$\left.\begin{array}{r} t_n\equiv b(hq_n-p_n) \mod p, \\ t_{n-1}\equiv b(p_{n-1}-hq_{n-1}) \mod p. \end{array}\right\} \tag{19}$$

If the sequence $a_0,a_1,a_2,\ldots$ from $\xi=[a_0;a_1,a_2,\ldots]$ is unbounded, there is an unbounded subset N of all positive integers such that, for certain integers $0\leq w_1<p$ and $0\leq w_2<p$, we have, for all $n\in N$:

$$a_n>2\sqrt{p}+1$$

and

$$\left.\begin{array}{r} hq_n-p_n\equiv w_1 \mod p, \\ p_{n-1}-hq_{n-1}\equiv w_2 \mod p. \end{array}\right\} \tag{20}$$

Without loss of generality, we may assume that all integers from N are *even*; in the case in which $a_0,a_1,a_2,\ldots$ has an unbounded subsequence only with odd indices, the arguments are the same apart from a change of sign in most of the subsequent formulas.

Moreover, if the sequence $a_0,a_1,a_2,\ldots$ is bounded, it is obvious that there is an unbounded subset N of all *even* positive integers satisfying (20) for all $n\in N$ with certain integers w_1 and w_2.

If it is $w_1=0$ or $w_2=0$, we have $p_n\equiv hq_n \mod p$ or $p_{n-1}\equiv hq_{n-1} \mod p$ for all $n\in N$; thus, the theorem is already proved in this case by taking the convergents $\frac{p_n}{q_n}$ or $\frac{p_{n-1}}{q_{n-1}}$ according to $w_1=0$ or $w_2=0$. The inequality (3) holds by (15); it remains to check the condition

$$p_n\equiv hq_n\not\equiv 0 \mod p \quad\text{or}\quad p_{n-1}\equiv hq_{n-1}\not\equiv 0 \mod p \quad (n\in N)$$

from the theorem. Assuming the contrary for $\frac{p_n}{q_n}$, we get

$$p_n\equiv 0\equiv hq_n \mod p.$$

From $h \not\equiv 0 \bmod p$ then follows $(p_n, q_n) \geq p$, a contradiction to a well-known fact. In the same way, one sees that $p_{n-1} \equiv hq_{n-1} \equiv 0 \bmod p$ for $n \in N$ is impossible.

It remains to treat the case in which $0 < w_1 < p$ and $0 < w_2 < p$. Now conditions (4) and (5) of Lemma 1 hold; hence, there are integers b, g_1, and g_2 satisfying (6), (7), and (8). By (8), (19), and (20), we may put

$$t_n = g_1, \quad t_{n-1} = g_2 \quad (n \in N). \tag{21}$$

We define, for $n \in N$,

$$\left.\begin{aligned} u_n &= p_n g_2 + p_{n-1} g_1, \\ v_n &= q_n g_2 + q_{n-1} g_1. \end{aligned}\right\} \tag{22}$$

By (17), for all $n \in N$, these integers u_n and v_n satisfy

$$u_n \equiv hb, \quad v_n \equiv b \bmod p, \tag{23}$$

and $b \not\equiv 0 \bmod p$ is a consequence of (8) and $0 < |g_1| < p$. In particular, we conclude that $u_n v_n \neq 0$.

Furthermore, we put, for $n \in N$,

$$\left.\begin{aligned} u_n(\alpha, \beta) &= p_n \alpha + p_{n-1} \beta, \\ v_n(\alpha, \beta) &= q_n \alpha + q_{n-1} \beta. \end{aligned}\right\} \tag{24}$$

This means that $u_n(g_2, g_1) = u_n$ and $v_n(g_2, g_1) = v_n$.

In the next step of our proof, we will follow Uchiyama [10]. From (7) we know that $g_1 \neq 0$ and $g_2 \neq 0$; therefore, we have to distinguish several cases according to the signs of g_1 and g_2. We always assume $n \in N$; in particular, we know that n is even.

Some additional arguments are necessary when $g_1 g_2 < 0$ to show that the sequence v_n from (22) is unbounded for certain subsets of N. This also fills the gap in Uchiyama's paper.

Case 1. $g_1 g_2 > 0$

From (16) it is clear that, for all even n, we have

$$\frac{p_n}{q_n} < \xi < \frac{p_{n-1}}{q_{n-1}}. \tag{25}$$

From this inequality and (22), we get

$$\frac{p_n}{q_n} < \frac{u_n}{v_n} < \frac{p_{n-1}}{q_{n-1}}. \tag{26}$$

The relationship between ξ and $\frac{u_n}{v_n}$ now gives occasion to consider two subcases according to $\xi < \frac{u_n}{v_n}$ or $\xi > \frac{u_n}{v_n}$:

Case 1.1. $\frac{p_n}{q_n} < \xi < \frac{u_n}{v_n}$ for infinitely many $n \in N_1 \subseteq N$

Let θ denote the sign of g_1 (resp. g_2). For integers $j \geq 0$, we define integers

$$\left.\begin{aligned} u_{n,j} &= u_n(g_2 + \theta jp, g_1) \overset{(22),(24)}{=} u_n + \theta pp_n j, \\ v_{n,j} &= v_n(g_2 + \theta jp, g_1) \overset{(22),(24)}{=} v_n + \theta pq_n j. \end{aligned}\right\} \tag{27}$$

Now we keep n fixed and, by straightforward computation, show that the fractions $\frac{u_{n,j}}{v_{n,j}}$ monotonically decrease as j increases, and that

$$\lim_{j\to\infty}\frac{u_{n,j}}{v_{n,j}}=\frac{p_n}{q_n}.$$

Hence, by the assumption of Case 1.1, there exists some unique integer $k\ge 1$ such that

$$\frac{u_{n,k}}{v_{n,k}}<\xi<\frac{u_{n,k-1}}{v_{n,k-1}}. \tag{28}$$

We also have

$$\frac{u_{n,k}}{v_{n,k}}<\frac{u_{n,k}+u_{n,k-1}}{v_{n,k}+v_{n,k-1}}<\frac{u_{n,k-1}}{v_{n,k-1}}, \tag{29}$$

since both inequalities in (29) are equivalent to

$$u_{n,k-1}v_{n,k}-v_{n,k-1}u_{n,k}>0;$$

this holds because

$$\begin{aligned}u_{n,k-1}v_{n,k}-v_{n,k-1}u_{n,k}&=\theta p(q_nu_n-p_nv_n)\\&\overset{(22)}{=}\theta p(p_{n-1}q_n-p_nq_{n-1})g_1\\&\overset{(16)}{=}(-1)^n\theta g_1p>0.\end{aligned} \tag{30}$$

Again two subcases arise from (28) and (29):

If we have

$$\frac{u_{n,k}}{v_{n,k}}<\xi<\frac{u_{n,k}+u_{n,k-1}}{v_{n,k}+v_{n,k-1}}<\frac{u_{n,k-1}}{v_{n,k-1}},$$

we assume that the following three inequalities hold simultaneously:

$$\xi-\frac{u_{n,k}}{v_{n,k}}\ge\frac{w}{\sqrt{5}v_{n,k}^2}, \tag{31}$$

$$\frac{u_{n,k}+u_{n,k-1}}{v_{n,k}+v_{n,k-1}}-\xi\ge\frac{w}{\sqrt{5}(v_{n,k}+v_{n,k-1})^2}, \tag{32}$$

$$\frac{u_{n,k-1}}{v_{n,k-1}}-\xi\ge\frac{w}{\sqrt{5}v_{n,k-1}^2}, \tag{33}$$

where $w=\theta g_1p$. We sum up (31) and (32) and also (31) and (33); after some calculations and application of (30), we get

$$\frac{1}{v_{n,k}(v_{n,k}+v_{n,k-1})}\ge\frac{1}{\sqrt{5}}\left(\frac{1}{v_{n,k}^2}+\frac{1}{(v_{n,k}+v_{n,k-1})^2}\right)$$

and

$$\frac{1}{v_{n,k}v_{n,k-1}}\ge\frac{1}{\sqrt{5}}\left(\frac{1}{v_{n,k}^2}+\frac{1}{v_{n,k-1}^2}\right).$$

From (22) and (27), from the definition of θ, and since $g_1g_2>0$, we know that $v_{n,k}v_{n,k-1}>0$ and, finally, all together contradict Lemma 2. Hence, at least one of the three inequalities—(31), (32), (33)—does not hold, and because each of the left-hand sides of these inequalities is positive, we have, for some

$$\frac{u}{v}\in\left\{\frac{u_{n,k}}{v_{n,k}},\frac{u_{n,k}+u_{n,k-1}}{v_{n,k}+v_{n,k-1}},\frac{u_{n,k-1}}{v_{n,k-1}}\right\},$$

$$\left|\xi-\frac{u}{v}\right|<\frac{w}{\sqrt{5}v^2}=\frac{p|g_1|}{\sqrt{5}v^2}\overset{(7)}{\leq}\frac{(1+\varepsilon)p^{3/2}}{\sqrt{5}v^2}. \tag{34}$$

But if ξ satisfies

$$\frac{u_{n,k}}{v_{n,k}}<\frac{u_{n,k}+u_{n,k-1}}{v_{n,k}+v_{n,k-1}}<\xi<\frac{u_{n,k-1}}{v_{n,k-1}},$$

we assume instead of (31), (32), (33),

$$\xi-\frac{u_{n,k}}{v_{n,k}}\geq\frac{w}{\sqrt{5}v_{n,k}^2}, \tag{35}$$

$$\xi-\frac{u_{n,k}+u_{n,k-1}}{v_{n,k}+v_{n,k-1}}\geq\frac{w}{\sqrt{5}(v_{n,k}+v_{n,k-1})^2}, \tag{36}$$

$$\frac{u_{n,k-1}}{v_{n,k-1}}-\xi\geq\frac{w}{\sqrt{5}v_{n,k-1}^2}. \tag{37}$$

Now we sum up (35) and (37), (36) and (37), which leads in the same way to a contradiction of Lemma 2. Hence, (34) holds in this subcase, too.

For every fraction $\frac{u}{v}$ satisfying (34), we know from (23) and (27) that either

$$u\equiv hv\equiv hb \mod p$$
or
$$u\equiv hv\equiv 2hb \mod p,$$

where $hb\not\equiv 0 \mod p$ implies $2hb\not\equiv 0 \mod p$ for all odd primes p. At last we note that $|v_n|$ tends to infinity for increasing $n\in N_1$; this follows from (22) and from $g_1g_2>0$. By (27) this means that, independently from k defined in (28), each of the numbers $|v_{n,k}|,|v_{n,k-1}|$, and $|v_{n,k}+v_{n,k-1}|$ tends to infinity for increasing $n\in N_1$. Thus, we have proved that there are infinitely many fractions $\frac{u}{v}$ satisfying (34), $u\equiv hv \mod p$ and, without loss of generality, $v>0$, provided the assumption of Case 1.1 holds for an unbounded subset N_1 of N.

Case 1.2. $\frac{u_n}{v_n}<\xi<\frac{p_{n-1}}{q_{n-1}}$ **for infinitely many** $n\in N_2\subseteq N$

In this case we define, for integers $j\geq 0$,

$$u_{n,j}=u_n(g_2,g_1+\theta jp)\overset{(22),(24)}{=}u_n+\theta pp_{n-1}j,$$

$$v_{n,j}=v_n(g_2,g_1+\theta jp)\overset{(22),(24)}{=}v_n+\theta pq_{n-1}j.$$

We proceed with similar arguments as in Case 1.1:

For any fixed n, the fractions $\frac{u_{n,j}}{v_{n,j}}$ monotonously increase with j; and from

$$\lim_{j\to\infty}\frac{u_{n,j}}{v_{n,j}}=\frac{p_{n-1}}{q_{n-1}}$$

we conclude that there is some unique integer $k\geq 1$ satisfying

$$\frac{u_{n,k-1}}{v_{n,k-1}}<\xi<\frac{u_{n,k}}{v_{n,k}}.$$

Again we consider the mediant $\frac{u_{n,k}+u_{n,k-1}}{v_{n,k}+v_{n,k-1}}$, which lies between $\frac{u_{n,k-1}}{v_{n,k-1}}$ and $\frac{u_{n,k}}{v_{n,k}}$, and distinguish two subcases according as ξ is greater or smaller than the mediant. Instead of (30), we now have

$$v_{n,k-1}u_{n,k}-u_{n,k-1}v_{n,k}=(-1)^n\theta g_2 p>0,$$

and as in Case 1.1 we get infinitely many fractions $\frac{u}{v}$ with

$$\left|\xi-\frac{u}{v}\right|<\frac{p|g_2|}{\sqrt{5}v^2}\overset{(7)}{\leq}\frac{(1+\varepsilon)p^{3/2}}{\sqrt{5}v^2}, \tag{38}$$

where $u\equiv hv\not\equiv 0 \mod p$, provided the assumption of Case 1.2 holds for an unbounded subset N_2 of N.

Case 2. $g_1>0,\ g_2<0$

Let $q_n>p$ and assume $v_n=0$. From $(q_n,q_{n-1})=1$ and (22), we have $q_n|g_1$, which is impossible because $0<|g_1|<p<q_n$. Hence, for sufficiently large n, we know that $v_n\neq 0$, and we distinguish two subcases according as $v_n>0$ or $v_n<0$. We may repeat all the arguments from Case 1 [with the exception of the infinity of the rationals $\frac{u}{v}$ in (34) or (38)]; we leave the details to the reader. We only state the definitions of the fractions $\frac{u_{n,j}}{v_{n,j}}$ corresponding to the subcases.

Case 2.1. $v_n>0$ **for infinitely many** $n\in N_3\subseteq N$

$$u_{n,j}=u_n(g_2+jp,g_1),$$
$$v_{n,j}=v_n(g_2+jp,g_1),\qquad \text{for } j\geq 0.$$

Case 2.2. $v_n<0$ **for infinitely many** $n\in N_4\subseteq N$

$$u_{n,j}=u_n(g_2,g_1-jp),$$
$$v_{n,j}=v_n(g_2,g_1-jp),\qquad \text{for } j\geq 0.$$

Case 3. $g_1<0,\ g_2>0$

Case 3.1. $v_n>0$ **for infinitely many** $n\in N_5\subseteq N$

$$u_{n,j}=u_n(g_2,g_1+jp),$$
$$v_{n,j}=v_n(g_2,g_1+jp),\qquad \text{for } j\geq 0.$$

Case 3.2. $v_n < 0$ for infinitely many $n \in N_6 \subseteq N$

$$u_{n,j} = u_n(g_2 - jp, g_1),$$
$$v_{n,j} = v_n(g_2 - jp, g_1), \qquad \text{for } j \geq 0.$$

It remains to show that in each of these four subcases there are *infinitely many* fractions $\frac{u}{v}$ satisfying (34) or (38). We treat only subcase 2.1; there are no essential differences in the other cases. In this last part of the proof of Theorem 1, we also complete some details in Uchiyama's paper [10].

It suffices to show that the sequence of integers $v_n = q_n g_2 + q_{n-1} g_1$ is not bounded if n takes all values from N_3. We assume the contrary, and from the assumptions of Case 2 we conclude that there is some positive real number C satisfying

$$0 < q_{n-1}|g_1| - q_n|g_2| \leq C \;\; (n \in N_3) \quad \text{or} \quad 0 < \left|\frac{g_1}{g_2}\right| - \frac{q_n}{q_{n-1}} \leq \frac{C}{|g_2| q_{n-1}} \to 0 \;\; \text{for } n \in N_3, n \to \infty$$

[note (7) and $v_n > 0$]. Hence:

$$\text{The sequence } \left(\frac{q_n}{q_{n-1}}\right)_{n \in N_3} \text{ tends to the positive rational number } \left|\frac{g_1}{g_2}\right|. \tag{39}$$

Let us first assume that the sequence $a_0, a_1, a_2, \ldots$ is unbounded; we recall from the definition of N that

$$n \in N_3 \subseteq N \Rightarrow a_n > 2\sqrt{p} + 1. \tag{40}$$

From the recurrence relation for q_n, we conclude

$$\frac{q_n}{q_{n-1}} = a_n + \frac{q_{n-2}}{q_{n-1}} \;\; (n \in N_3); \tag{41}$$

and by $q_{n-2} < q_{n-1}$ it follows for all sufficiently large integers $n \in N_3$ that

$$a_n = \left[\frac{q_n}{q_{n-1}}\right] \leq \frac{q_n}{q_{n-1}} \overset{(39)}{\leq} \left|\frac{g_1}{g_2}\right| + 1 \leq |g_1| + 1 \overset{(7)}{\leq} 2\sqrt{p} + 1,$$

which contradicts (40).

Now we treat the more interesting case where the sequence $a_0, a_1, a_2, \ldots$ is bounded. In what follows, we assume that $n \in N_3$ is sufficiently large. We denote the continued fraction expansion of $\left|\frac{g_1}{g_2}\right|$ for some integer r and $c_r > 1$ by*

$$\left|\frac{g_1}{g_2}\right| = [c_0; c_1, c_2, \ldots, c_r].$$

It is a well-known fact from the elementary theory of continued fractions that

$$\frac{q_n}{q_{n-1}} = [a_n; a_{n-1}, \ldots, a_2, a_1]. \tag{42}$$

---

* In the case $r = 0$, $c_0 = 1$, $g_1 = -g_2$, it is clear that $|v_n| = |g_1|(a_n q_{n-1} + q_{n-2} - q_{n-1}) \geq |g_1| q_{n-2}$ tends to infinity.

On the other hand we have, from (39) for all sufficiently large integers $n \in N_3$,

$$\frac{q_n}{q_{n-1}} = [c_0; c_1, \ldots, c_r + \delta(n)],$$

where

$$0 \neq \delta(n) \to 0 \text{ for } n \in N_3, n \to \infty. \tag{43}$$

For $0 < \delta(n) < 1$ we have, from (42),

$$a_{n-r} = c_r,$$
$$a_{n-r-1} = \left[\frac{1}{\delta(n)}\right]. \tag{44}$$

$-1/2 < \delta(n) < 0$ implies

$$a_{n-r} = c_r - 1,$$
$$a_{n-r-1} = \left[\frac{1}{1+\delta(n)}\right] = 1,$$
$$a_{n-r-2} = \left[\frac{1}{\frac{1}{1+\delta(n)} - 1}\right] = \left[-\frac{1}{\delta(n)} - 1\right]. \tag{45}$$

By (43), (44), and (45), a certain unbounded subsequence of $a_0, a_1, a_2, \ldots$ is given, a contradiction to our assumption.

The proof of Theorem 1 is now complete.

4. CONCLUDING REMARKS

The application of Lemma 2 in the proof of Theorem 1 will lead to some nice results by the way, if we put $s = 2$ instead of $s = p$ in (17). The following result is, in some sense, a supplement of Scott's and Robinson's theorems (see [9] and [8]).

Theorem 2: For any irrational number ξ, there are infinitely many pairs of integers u_1 and $v_1 > 0$, respectively u_2 and $v_2 > 0$, satisfying

(i) $$\left|\xi - \frac{u_1}{v_1}\right| \leq \frac{2}{\sqrt{5}v_1^2} \quad \text{and} \quad u_1 \equiv v_1 \bmod 2;$$

(ii) $$\left|\xi - \frac{u_2}{v_2}\right| \leq \frac{2}{\sqrt{5}v_2^2} \quad \text{and} \quad v_2 \equiv 0 \bmod 2, \text{ respectively.}$$

To prove (i), put $a = 1$ and $b = 1$ in (17); for (ii), let $a = 1$ and $b = 0$. It is obvious that we do not need Lemma 1.

Now let us consider such a real number ξ where $a_n \equiv 0 \bmod 2$ for all $n \geq 0$ in the continued fraction expansion of ξ. From the recurrence relations, it can easily be seen that this implies

$$p_n + q_n \equiv 1 \bmod 2 \quad (n \geq 1).$$

For instance $\xi = 1+\sqrt{2} = [2; \overline{2}]$ belongs to these numbers. We derive the following corollary from Theorem 2(i).

Corollary 1: There is an uncountable set of real numbers such that, for every number ξ from this set, there are infinitely many Dirichlet-approximants $\frac{u}{v}$ satisfying

$$\left|\xi - \frac{u}{v}\right| \le \frac{2}{\sqrt{5}v^2},$$

such that no fraction $\frac{u}{v}$ belongs to $\left\{\frac{p_n}{q_n} : n \ge 1\right\}$.

To appreciate this corollary, we refer to Theorem 184 in [2], which states that

$$\frac{u}{v} \in \left\{\frac{p_n}{q_n} : n \ge 1\right\}, \text{ if } \left|\xi - \frac{u}{v}\right| < \frac{1}{2v^2}.$$

REFERENCES

1. L. C. Eggan. "On Diophantine Approximations." *Trans. Amer. Math. Soc.* **99** (1961):102-17.
2. G. H. Hardy & E. M. Wright. *An Introduction to the Theory of Numbers*. 5th ed. Oxford: Clarendon Press, 1984.
3. S. Hartman. "Sur une condition supplementaire dans les approximations diophantiques." *Colloq. Math.* **2** (1951):48-51.
4. J. F. Koksma. "Sur l'approximation des nombres irrationnels sous une condition supplementaire." *Simon Stevin* **28** (1951):199-202.
5. L. Kuipers & B. Meulenbeld. "Some Properties of Continued Fractions." *Acta Math.* **87** (1952):1-12.
6. A. Oppenheim. "Rational Approximations to Irrationals." *Bull. Amer. Math. Soc.* **47** (1941): 602-04.
7. G. J. Rieger. *Zahlentheorie*. Studia Mathematica **29**. Göttingen: Vandenhoeck u. Ruprecht, 1976.
8. R. M. Robinson. "The Approximation of Irrational Numbers by Fractions with Odd or Even Terms." *Duke Math. J.* **7** (1940):354-59.
9. W. T. Scott. "Approximation to Real Irrationals by Certain Classes of Rational Fractions." *Bull. Amer. Math. Soc.* **46** (1940):124-29.
10. S. Uchiyama. "On Rational Approximations to Irrational Numbers." *Tsukuba J. Math.* **4** (1980):1-7.

AMS Classification Numbers: 11J04, 11J70