# DIAGONALIZATION OF THE BINOMIAL MATRIX

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## 1. INTRODUCTION

The results of this paper assume a familiarity with linear algebra. A good reference for the results assumed here is [1].

As is well known, the Fibonacci numbers may be generated in the following manner. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Then

$$A^h \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} F_{h+1} \\ F_h \end{bmatrix}.$$

If we diagonalize $A$ as

$$A = BDB^{-1} = \begin{bmatrix} \gamma & -1/\gamma \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \gamma & 0 \\ 0 & -1/\gamma \end{bmatrix} \begin{bmatrix} \gamma & -1/\gamma \\ 1 & 1 \end{bmatrix}^{-1},$$

where $\gamma = \frac{1+\sqrt{5}}{2}$ is the golden ratio, then from

$$\begin{bmatrix} F_{h+1} \\ F_h \end{bmatrix} = A^h \begin{bmatrix} 1 \\ 0 \end{bmatrix} = BD^hB^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

one obtains the formula

$$F_h = \frac{\gamma^h - (-1/\gamma)^h}{\sqrt{5}}. \tag{1.1}$$

More generally, if $f(x) = x^m - s_1x^{m-1} - \cdots - s_m$ is a polynomial with distinct roots $\alpha_i$, and $C$ is the companion matrix of $f(x)$,

$$C = \begin{bmatrix} s_1 & s_2 & \cdots & s_{m-1} & s_m \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix},$$

then $v_h = C^h v_0$ generates the recurrence sequence with initial values given by $v_0$ and recurrence polynomial $f(x)$. Again, we can diagonalize $C = BDB^{-1}$ and obtain the formula

$$a_h = \sum A_i \alpha_i^h, \tag{1.2}$$

for some $A_i \in \mathbb{C}$.

In fact, there is nothing special about companion matrices here. If $M$ is any square matrix over $\mathbb{Z}$ (say) and $v_h = M^h v_0$, then, as we shall prove in the next section, each component of $v_h$ is a recurrence sequence with recurrence polynomial equal to the characteristic polynomial of $M$.

Now let us examine some generalizations of the relation above for the Fibonacci numbers. One way to generalize the matrix $A$ above is to the binomial matrix. For example, consider

$$A_3 = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}; \quad v_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

By following the above method, we find that the characteristic polynomial of $A_3$ is $x^3 - 2x^2 - 2x + 1$; the eigenvalues of $A_3$ are $1/\gamma^2, -1, \gamma^2$; and

$$A_3^h v_0 = \begin{bmatrix} F_{h+1}^2 \\ F_h F_{h+1} \\ F_h^2 \end{bmatrix}. \tag{1.3}$$

We prove in this article a generalization of this observation. We find that the eigenvalues of the $n$-by-$n$ binomial matrix are powers of the golden ratio. As a consequence, we shall derive the generalization of (1.3) above. Moreover, we show how explicitly to diagonalize this binomial matrix, and we give recurrence relations for the characteristic polynomials.

More precisely, let $\gamma = \frac{1+\sqrt{5}}{2}$ be the golden ratio. Let $A_n = [a_{i,j}]$ be the "inverted" (or upside-down) binomial matrix (Pascal's triangle):

$$a_{ij} = \begin{cases} 0 & \text{if } i+j > n+1, \\ \binom{n-i}{j-1} & \text{otherwise.} \end{cases}$$

Let $W_n = \{\gamma^{n-1-i}(-1/\gamma)^i\}_{i=0}^{n-1}$ and let $Q_n(x) = \prod_{w \in W_n}(x-w)$. Let $D_n$ be the diagonal matrix whose diagonal entries are the elements of $W_n$ listed in decreasing order according to size of the absolute value. Let $E_n$ be the eigenvector matrix of $A_n$ with column vectors listed in decreasing order of absolute value of the corresponding eigenvalues, and with its columns scaled so that the bottom row is all 1's. So, for example, for $n = 5$ we have

$$A_5 = \begin{bmatrix} 1 & 4 & 6 & 4 & 1 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad D_5 = \begin{bmatrix} \gamma^4 & 0 & 0 & 0 & 0 \\ 0 & -\gamma^2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\gamma^{-2} & 0 \\ 0 & 0 & 0 & 0 & \gamma^{-4} \end{bmatrix},$$

$$E_5 = \begin{bmatrix} \gamma^4 & -\gamma^2 & 1 & -\gamma^{-2} & \gamma^{-4} \\ \gamma^3 & -\gamma^{-1}/4 & -1/2 & \gamma/4 & -\gamma^{-3} \\ \gamma^2 & \gamma/2 & -1/6 & -\gamma^{-1}/2 & \gamma^{-2} \\ \gamma & \gamma^3/4 & 1/2 & -\gamma^{-3}/4 & -\gamma^{-1} \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

The main result follows.

***Theorem 1.4:*** The eigenvalues of $A_n$ are exactly the set of values in $W_n$, so that the characteristic polynomial of $A_n$ is $Q_n(x)$. Moreover, an explicit recursive method can be given for generating $E_n$ and $E_n^{-1}$ so that we can diagonalize $A_n$ explicitly as

$$E_n^{-1}A_nE_n = D_n.$$

In addition, the coefficients of the characteristic polynomial of $A_n$ can be generated recursively.

In Section 2 we present some background on recurrence sequences and derive a simple self-contained proof of the first statement (Theorem 2.8). In Section 3 we give recurrence relations for the characteristic polynomial array (Definition 3.4, Proposition 3.5, and Corollary 3.11) and in Section 4 we give an explicit diagonalization of $A_n$ (Theorem 4.3). As a consequence, we obtain a second proof that the characteristic polynomial of $A_n$ is $Q_n(x)$. In Section 5 we give a generalization (Theorem 5.10). This approach demonstrates the explicit recursive method (Corollaries 5.8 and 5.9) for generating the eigenvector matrices. As a consequence of this approach, we obtain a third proof that the characteristic polynomial of $A_n$ is $Q_n(x)$. However, slightly more algebra is required for this approach.

The first proof is based on elementary facts from vector recurrences, for which we provide a quick review. We give an overview of the second proof. We define recursively an array of numbers $b_{n,m}$ (Definition 3.4). From the $n^{\text{th}}$ row of this array of numbers $b_{n,m}$, we form a polynomial $P_n(x)$. We show inductively that the roots of $P_n(x)$ form the set $W_n$, whence $P_n(x) = Q_n(x)$. Finally, we demonstrate that the companion matrix of $P_n(x)$ is similar to $A_n$, giving our result, since similar matrices have the same eigenvalues. The similarity computation requires the auxiliary matrices that we define in Section 4.

## 2. REVIEW OF RECURRENCE SEQUENCES

We present a review of recurrence sequences and, as a consequence, obtain a quick proof of the first statement of Theorem 1.4. Moreover, we find an interesting characterization of recurrence sequences generated by $Q_n(X)$ (Theorem 2.8) using some of the results developed in later sections. See [3] for generalities regarding recurrence sequences.

***Definition 2.1:*** A sequence $(a_h)$ satisfying a linear recursion

$$a_h = \sum_{k=1}^{n} s_k a_{h-k}$$

is called a (linear) *recurrence sequence*. We call the polynomial $x^n - \sum_{k=1}^{n} s_k x^{n-k}$ the *recurrence polynomial* for $(a_h)$, and we say it generates $(a_h)$. We call $(a_h)$ *degenerate* if it is also generated by a polynomial of smaller degree.

If $f(x)$ has $m$ roots $\alpha_k$, then, as in (1.2), it is easy to show that

$$a_h = \sum_{k=1}^{m} A_k(h)\alpha_k^h, \tag{2.2}$$

where the $A_k(h)$ is a polynomial whose degree is the multiplicity of $\alpha_k$ in $f(x)$. Moreover, any such *generalized power sum* is a recurrence sequence with recurrence polynomial $f(x)$. Hence, it follows that the set of all recurrence sequences with recurrence polynomial $f(x)$ is a vector space of dimension $n$.

We shall make use of the following proposition in Section 4 below.

***Proposition 2.3:*** Let $x_h$ be a recurrence sequence of degree $s$ with recurrence polynomial $p(x) = \prod_{k=1}^{s}(x-\alpha_k)$ with the $\alpha_i$ distinct and let $y_h$ be a recurrence sequence of degree $t$ with recurrence polynomial $q(x) = \prod_{\ell=1}^{t}(x-\beta_\ell)$ with the $\beta_\ell$ distinct. Let $W$ be the distinct set of numbers of the form $\alpha_k\beta_\ell$ with $w = |W|$. Then the sequence $x_h y_h$ is a recurrence sequence of degree $w$ with recurrence polynomial $\prod_{\lambda\in W}(x-\lambda)$.

***Proof:*** The vector space of sequences with recurrence polynomial $p(x)$ is spanned by the sequences $\alpha_k^h$ for $k = 1, \ldots, s$. Thus, we can write $x_h = \sum_{k=1}^{s} u_k\alpha_i^h$ for some $u_k$. Similarly, we can write $y_h = \sum_{\ell=1}^{t} v_\ell\beta_\ell^h$ for some $v_\ell$. Multiplying yields

$$x_h y_h = \sum_{k,\ell} u_k v_\ell (\alpha_k\beta_\ell)^h.$$

Thus, $x_h y_h$ is in the span of the sequences $\lambda^h$ for $\lambda \in W$ and, hence, has recurrence polynomial as above. □

It is easy to characterize the space of sequences generated by a polynomial.

***Proposition 2.4:*** The sequence $(a_h)$ is a nondegenerate recurrence sequence generated by $f(x)$ of degree $n$, if and only if the matrix

$$A = \begin{bmatrix} a_{n-1} & a_n & \cdots & a_{2n-1} \\ a_{n-2} & a_{n-1} & \cdots & a_{2n-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_0 & a_1 & \cdots & a_{n-1} \end{bmatrix}$$

is invertible. In this case, the $n$ sequences $(a_{h+k})_{k=0}^{n-1}$ generate the space of recurrence sequences generated by $f(x)$.

***Proof:*** If $A$ had a nontrivial element in its kernel, then so would

$$C^h A = \begin{bmatrix} a_{h+n-1} & a_{h+n} & \cdots & a_{h+2n-1} \\ a_{h+n-2} & a_{h+n-1} & \cdots & a_{h+2n-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_h & a_{h+1} & \cdots & a_{h+n-1} \end{bmatrix},$$

where $C$ is the companion matrix for $f(x)$. This is true if and only if $(a_h)$ is a degenerate recurrence sequence. □

Next, we consider recurrence sequences that arise from matrices. This generalization is quite simple.

***Definition 2.5:*** Let $M$ be an $n$-by-$n$ matrix and let $v_0$ be an $n$-dimensional column vector. The sequence of vectors $(v_h)$ defined by $v_h = M^h v_0$ is called a *vector recurrence sequence*.

If $M$ is the companion matrix for $f(x)$,

$$M = C = \begin{bmatrix} s_1 & s_2 & \cdots & s_{m-1} & s_m \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix},$$

then our situation is closely related to recurrence sequences. Let

$$v_0 = \begin{bmatrix} a_{n-1} \\ \vdots \\ a_0 \end{bmatrix},$$

and let $(a_h)$ be the corresponding recurrence sequences generated by $f(x)$. Then

$$v_h = C^h v_0 = \begin{bmatrix} a_{n+h-1} \\ \vdots \\ a_h \end{bmatrix}.$$

Even in a more general case, this picture is only altered with a change of basis.

***Proposition 2.6:*** Let $M$ be an $n$-by-$n$ matrix with characteristic polynomial $f(x)$. Suppose further that $f(x)$ is actually the minimal polynomial of $M$, so that we have the similarity relation $M = BCB^{-1}$, where $C$ is the companion matrix of $f(x)$. Let $(v_h)$ be the vector recurrence sequence generated by $M$ with initial value $v_0$. Then the $i^{\text{th}}$ component of $(v_h)$ forms a recurrence sequence $(v_h^i)$ with recurrence polynomial $f(x)$. Moreover, the recurrence sequence generated by $f(x)$ with initial values given by $B^{-1}v_0$ is nondegenerate if and only if the $n$-by-$n$ matrix $[v_0 \ \cdots \ v_{n-1}]$ is invertible. Hence, in this case, the $(v_n^i)$ form a basis of the space of recurrence sequences generated by $f(x)$.

***Remark:*** The condition that $f(x)$ is the minimal polynomial of $M$ is not necessary; however, the statement becomes more complicated and the conclusion weaker.

***Proof:*** Using the similarity relation, we find that

$$C \cdot \left[(B^{-1}v_0) \ \cdots \ (B^{-1}v_{n-1})\right] = \left[(B^{-1}v_1) \ \cdots \ (B^{-1}v_n)\right].$$

Thus, $(B^{-1}v_h)$ is a vector recurrence sequence for the matrix $C$. In other words,

$$B^{-1}[v_0 \ \cdots \ v_{n-1}] = A = \begin{bmatrix} a_{n-1} & a_n & \cdots & a_{2n-1} \\ a_{n-2} & a_{n-1} & \cdots & a_{2n-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_0 & a_1 & \cdots & a_{n-1} \end{bmatrix},$$

where $(a_h)$ is a recurrence sequence generated by $f(x)$. Thus,

$$M^h v_0 = BC^h(B^{-1}v_0) = B\begin{bmatrix} a_{n+h-1} \\ \vdots \\ a_h \end{bmatrix}.$$

This implies that the $i^{\text{th}}$ component $(v_h^i)$ is a linear combination of recurrence sequences generated by $f(x)$; hence, $(v_h^i)$ itself is generated by $f(x)$.

The last statement then follows by applying Proposition 2.4 to $A = B^{-1}[v_0 \cdots v_{n-1}]$, and noting that $B$ is invertible. □

We apply the above development to our binomial matrices. We shall consider the sequences $(F_{h+1}^{n-i}F_h^{i-1})$ for $1 \le i \le n$ as column vectors for fixed $h$.

***Proposition 2.7:*** For all $h \ge 0$, $A_n[F_{h+1}^{n-i}F_h^{i-1}] = [F_{h+2}^{n-i}F_{h+1}^{i-1}]$. In other words, $(F_{h+1}^{n-i}F_h^{i-1})$ is a recurrence sequence generated by the characteristic polynomial of $A_n$ for all $i$, $1 \le i \le n$.

***Proof:*** This follows trivially by induction. One must only check the top entry, which follows from the relation

$$\sum_{j=1}^{n}\binom{n-1}{j-1}F_{h+1}^{n-j}F_h^{j-1} = (F_{h+1}+F_h)^{n-1} = F_{h+2}^{n-1}. \ \square$$

Now we use (1.1):

$$F_{h+1}^{n-i}F_h^{i-1} = \left(\frac{\gamma^{h+1}-(-1/\gamma)^{h+1}}{\sqrt{5}}\right)^{n-i}\left(\frac{\gamma^{h}-(-1/\gamma)^{h}}{\sqrt{5}}\right)^{i-1} = \sum_{w\in W_n} A_w w^h.$$

Thus, by (2.2), the polynomial $Q_n(x) = \prod_{w\in W_n}(x-w)$ generates each of the sequences $(F_{h+1}^{n-i}F_h^{i-1})$. Clearly, if we can show that $(F_{h+1}^{n-i}F_h^{i-1})$ is nondegenerate of degree $n$, then we must have

$$\text{char poly}(A_n) = Q_n(x).$$

***Theorem 2.8:*** The eigenvalues of $A_n$ are $W_n$; hence, the characteristic polynomial of $A_n$ is $Q_n(x)$. Moreover, the sequences $(F_{h+1}^{n-i}F_h^{i-1})_{i=1}^{n}$ form a basis for the space of recurrence sequences generated by $Q_n(x)$.

***Proof:*** In light of the above, it only remains to show that the matrix $[F_{j+1}^{n-i}F_j^{i-1}]$ is invertible. If we scale the $j^{\text{th}}$ column by dividing by $F_{j+1}^{n-1}$, then we obtain the Vandermonde matrix $[(F_j / F_{j+1})^{i-1}]$, and as is well known, this determinant is nonzero. □

## 3. THE CHARACTERISTIC POLYNOMIAL

We set out some well-known (and easily proved) facts about the Fibonacci and Lucas numbers to refer to later.

$$\gamma^h = \frac{L_h + F_h\sqrt{5}}{2}. \tag{3.1}$$

$$\gamma^{-h} = (-1)^h\frac{L_h - F_h\sqrt{5}}{2}. \tag{3.2}$$

$$F_hL_k + F_kL_h = 2F_{h+k}. \tag{3.3}$$

We shall see that the following array of numbers gives the coefficients of the characteristic polynomial of $A_n$.

***Definition 3.4:*** Define the array of numbers $b_{n,m}$ for $n, m \ge 0$ as follows. Let $b_{n,0} = 1$ for all $n \ge 0$ and let $b_{n,m} = 0$ for $m > n$. For $0 < m \le n$, we define $b_{n,m}$ recursively by

$$b_{n,m} = b_{n-1,m-1}\frac{F_n}{F_m}(-1)^m.$$

***Proposition 3.5:*** For $m \le n$, we have

$$|b_{n,m}| = \frac{F_n F_{n-1} \cdots F_{n-m+1}}{F_m \cdots F_1}.$$

Moreover, $|b_{n,m}| = |b_{n,n-m}|$, and we have the relation

$$b_{n,m} = b_{n,m-1}\frac{F_{n-m+1}}{F_m}(-1)^m.$$

The proof is obvious. The first several rows of the $b_{n,m}$ array are given in Table 3.6, where $n$ indexes the rows and $m$ indexes the columns.

**TABLE 3.6. Coefficients of the Characteristic Polynomial**

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | −1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 1 | −1 | −1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | 1 | −2 | −2 | 1 | 0 | 0 | 0 | 0 | 0 |
| 4 | 1 | −3 | −6 | 3 | 1 | 0 | 0 | 0 | 0 |
| 5 | 1 | −5 | −15 | 15 | 5 | −1 | 0 | 0 | 0 |
| 6 | 1 | −8 | −40 | 60 | 40 | −8 | −1 | 0 | 0 |
| 7 | 1 | −13 | −104 | 260 | 260 | −104 | −13 | 1 | 0 |
| 8 | 1 | −21 | −273 | 1092 | 1820 | −1092 | −273 | 21 | 1 |

***Definition 3.7:*** Let $P_n(x) = \sum_{j=0}^{n} b_{n,n-j}x^j$. The first few $P_n(x)$ (which can be read from Table 3.6) and $W_n$ (defined in Section 1) are:

$$\begin{aligned}
P_1(x) &= x-1 & W_1 &= \{1\}\\
P_2(x) &= x^2-x-1 & W_2 &= \{\gamma, -\gamma^{-1}\}\\
P_3(x) &= x^3-2x^2-2x+1 & W_3 &= \{\gamma^2, -1, \gamma^{-2}\}\\
P_4(x) &= x^4-3x^3-6x^2+3x+1 & W_4 &= \{\gamma^3, -\gamma, \gamma^{-1}, -\gamma^{-3}\}
\end{aligned}$$

We note that the $(n-1)$ column of the $b_{n,m}$ array is just the coefficients of the formal power series $(-1)^n / P_n((-1)^{n-1}x)$. This is equivalent to

$$\sum_{k=0}^{n}(-1)^{k(n-1)}b_{n,k}b_{k+j,n-1} = 0$$

for all $j \ge n-1$. Although we do not use this fact here, we record it as Corollary 3.11 to Theorem 3.8.

***Theorem 3.8:*** The set of roots of the polynomial $P_n(x)$ is exactly $W_n$: $P_n(x) = Q_n(x)$.

***Proof:*** We use induction. Since $W_n = (-1/\gamma)W_{n-1} \cup \{\gamma^{n-1}\}$, we have the relation

$$Q_n(x) = \frac{1}{(-\gamma)^{n-1}}P_{n-1}(-\gamma x)\cdot(x-\gamma^{n-1}). \tag{3.9}$$

We rewrite this as

$$Q_n(x) = (x + (-1)^n(-\gamma)^{n-1})\sum_{j=0}^{n-1} b_{n-1,\, n-1-j}(-\gamma)^{j-n+1}x^j$$

$$= x^n + (-1)^n b_{n-1,\, n-1} + \sum_{j=1}^{n-1} [b_{n-1,\, n-j}(-\gamma)^{j-n} + (-1)^n b_{n-1,\, n-1-j}(-\gamma)^j]x^j.$$

Thus, we need to show the relation

$$b_{n,\, n-j} = b_{n-1,\, n-j}(-\gamma)^{j-n} + (-1)^n b_{n-1,\, n-1-j}(-\gamma)^j. \tag{3.10}$$

By equations (3.1) and (3.2), this can be written as

$$b_{n,\, n-j} = b_{n-1,\, n-j}\left(\frac{L_{n-j} - F_{n-j}\sqrt{5}}{2}\right) + (-1)^{n+j} b_{n-1,\, n-j-1}\left(\frac{L_j + F_j\sqrt{5}}{2}\right).$$

By Proposition 3.5, this becomes

$$b_{n,\, n-j} = (-1)^{n+j} b_{n-1,\, n-j-1}\left[\frac{F_j}{F_{n-j}}\left(\frac{L_{n-j} - F_{n-j}\sqrt{5}}{2}\right) + \frac{L_j + F_j\sqrt{5}}{2}\right]$$

$$= (-1)^{n+j} b_{n-1,\, n-j-1}\left[\frac{F_j L_{n-j} + F_{n-j} L_j}{2F_{n-j}}\right].$$

By equation (3.3) above, this simplifies to

$$b_{n,\, n-j} = (-1)^{n+j} b_{n-1,\, n-j-1}\frac{F_n}{F_{n-j}},$$

which follows from Definition 3.4. □

***Corollary 3.11:*** The $(n-1)$ column of the array $b_{n,m}$ forms the coefficients of the formal power series $(-1)^n / P_n((-1)^{n-1}x)$. More precisely, we have

$$\frac{(-1)^n}{P_n((-1)^{n-1}x)} = \sum_{k=0}^{\infty} b_{k+n-1,\, n-1}x^k.$$

***Proof:*** Repeated application of (3.10) gives the following:

$$b_{n,\, m} = \gamma^{n+1}\sum_{k=1}^{n-m+1} (-1)^{km}\gamma^{-k(m+1)} b_{n-k,m-1}.$$

By changing the order of summation and relabeling the indices, this is equivalent to

$$b_{k+n-1,\, n-1} = (-1)^{n-1}\sum_{s=0}^{k} b_{s+n-2,\, n-2}\gamma^s(-\gamma)^{(k-s)(1-n)}.$$

But this just expresses the power series identity

$$\sum_{k=0}^{\infty} b_{k+n-1,\,n-1} x^k = (-1)^{n-1}\left(\sum_{s=0}^{\infty} b_{s+n-2,\,n-2}(\gamma x)^s\right)\left(\sum_{t=0}^{\infty}\left(\frac{x}{(-\gamma)^{n-1}}\right)^t\right). \tag{3.12}$$

Now, applying induction, the inverse of the right-hand side of (3.12) is just

$$(-1)^{n-1}\left(\frac{P_{n-1}((-1)^{n-2}\gamma x)}{(-1)^{n-1}}\right)\left(1-\frac{x}{(-\gamma)^{n-1}}\right)=\frac{(-1)^n}{(-\gamma)^{n-1}}P_{n-1}((-1)^{n-2}\gamma x)((-1)^{n-1}x-\gamma^{n-1}).$$

Substituting $y=(-1)^{n-1}x$, we have

$$\frac{(-1)^n}{(-\gamma)^{n-1}}P_{n-1}(-\gamma y)(y-\gamma^{n-1})=(-1)^n P_n(y),$$

by (3.9) and Theorem 3.8. Thus, as required, the left-hand side of (3.12) is

$$\sum_{k=0}^{\infty} b_{k+n-1,\,n-1}x^k = \frac{(-1)^n}{P_n((-1)^{n-1}x)}. \quad \square$$

## 4. EXPLICIT DIAGONALIZATION

We define the following integral matrices that will be used to diagonalize $A_n$ in Theorem 4.3 explicitly.

***Definition 4.1:*** Let $n>1$. Let $C_n$ be the companion matrix for $P_n(x)$, $C_n=[c_{ij}]$, where

$$\begin{cases} c_{i,i+1}=1 & \text{for } i=1,\dots,n-1,\\ c_{n,j}=-b_{n,n+1-j} & \text{for } j=1,\dots,n,\\ c_{i,j}=0 & \text{otherwise.}\end{cases}$$

Let $R_n=[r_{ij}]$, where $r_{ij}=\binom{n-1}{j-1}F_{i-2}^{j-1}F_{i-1}^{n-j}$. Let $M_n=[m_{ij}]$, where $m_{ij}=\binom{n-1}{j-1}F_{i-1}^{j-1}F_i^{n-j}$.

Observe that the $i^{\text{th}}$ row of $R_n$ gives the terms in $(F_{i-2}+F_{i-1})^{n-1}$ and that the $i^{\text{th}}$ row of $M_n$ gives the terms in $(F_{i-1}+F_i)^{n-1}$. These matrices will be used in Theorem 4.3 to prove that $A_n$ is similar to $C_n$. The matrix $R_n$ arose originally by observing the relation $R_nE_n=V_n$ (see Definition 4.2). From here, it is natural to bring in the companion matrix $C_n$, since $V_n$ is the eigenvector matrix for $C_n$.

We illustrate Definition 4.1 for $n=5$:

$$C_5=\begin{bmatrix}0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\1&-5&-15&15&5\end{bmatrix},\qquad R_5=\begin{bmatrix}0&0&0&0&1\\1&0&0&0&0\\1&4&6&4&1\\16&32&24&8&1\\81&216&216&96&16\end{bmatrix},$$

$$M_5=\begin{bmatrix}1&0&0&0&0\\1&4&6&4&1\\16&32&24&8&1\\81&216&216&96&16\\625&1500&1350&540&81\end{bmatrix}.$$

***Definition 4.2:*** For each $n>1$, let $V_n$ be the Vandermonde matrix which is the eigenvector matrix for $C_n$ with eigenvectors listed in decreasing order of the absolute values of the corresponding eigenvalues.

Thus, for example, we have

$$V_5=\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \gamma^4 & -\gamma^2 & 1 & -\gamma^{-2} & \gamma^{-4} \\ \gamma^8 & \gamma^4 & 1 & \gamma^{-4} & \gamma^{-8} \\ \gamma^{12} & -\gamma^6 & 1 & -\gamma^{-6} & \gamma^{-12} \\ \gamma^{16} & \gamma^8 & 1 & \gamma^{-8} & \gamma^{-16} \end{bmatrix}.$$

***Theorem 4.3:*** For all $n$, we have the relation $M_n=C_nR_n=R_nA_n$. Moreover, $P_n(x)$ is the characteristic polynomial of $A_n$, $R_n$ is invertible, and the eigenvector matrix of $A_n$ is given by

$$E_n=R_n^{-1}V_n.$$

***Proof:*** Multiplying the first $n-1$ rows of $C_n$ by $R_n$ clearly gives the first $n-1$ rows of $M_n$. For the last row, for each $j$, $1\le j\le n$, we must show the relation

$$\sum_{k=1}^{n}-b_{n,\,n+1-k}F_{k-2}^{j-1}F_{k-1}^{n-j}=F_{n-1}^{j-1}F_n^{n-j}. \tag{4.4}$$

Now $P_2(x)$ is the recurrence polynomial for the sequence $F_k$ (and, hence, for $F_{k-1}$). Thus, using the fact that $W_uW_v=W_{u+v-1}$, we can apply Proposition 2.3 repeatedly to find that $P_m(x)$ is the recurrence polynomial for the sequence whose $h^{\text{th}}$ entry is a product of $m-1$ factors, each chosen from the set $\{F_h,F_{h-1}\}$. In other words, $P_m(x)$ is the recurrence polynomial for the sequence $F_{h-1}^{j-1}F_h^{m-j}$ for any $j$, $1\le j\le m$. Explicitly, this means that

$$\sum_{k=1}^{m}-b_{m,\,m+1-k}F_{r+k-m-2}^{j-1}F_{r+k-m-1}^{m-j}=F_{r-1}^{j-1}F_r^{m-j}.$$

Equation (4.4) now follows, since it is just this same recurrence relation for $m=n$ at the $r=n$ term. This proves $M_n=C_nR_n$.

To prove $M_n=R_nA_n$ it is equivalent to show that, for all $i,j$ with $1\le i,j\le n$, we have

$$\sum_{k=0}^{n-j}\binom{n-1}{k}F_{i-2}^{k}F_{i-1}^{n-k-1}\cdot\binom{n-k-1}{j-1}=\binom{n-1}{j-1}F_{i-1}^{j-1}F_i^{n-j}.$$

Combining the binomials and dividing by $F_{i-1}^{j-1}$, this is equivalent to showing

$$\sum_{k=0}^{n-j}\binom{n-j}{k}F_{i-2}^{k}F_{i-1}^{n-k-j}=F_i^{n-j}.$$

But, by the binomial theorem, this is just $(F_{i-2}+F_{i-1})^{n-j}=F_i^{n-j}$, which is just the Fibonacci recursion. This proves that $C_nR_n=R_nA_n$.

The fact that $R_n$ is invertible was actually proved previously in the proof of Theorem 2.8; again, we can scale $R_n$ to obtain a Vandermonde matrix which has a nonzero determinant. Hence, $C_n$ and $A_n$ satisfy the similarity relation $A_n=R_n^{-1}C_nR_n$. Thus, they have the same characteristic polynomial $P_n(x)$. Since $V_n$ is the eigenvector matrix for $C_n$, the similarity relation shows that $R_n^{-1}V_n$ is the eigenvector matrix for $A_n$. □

## 5. A GENERALIZATION

In this section we give an alternative development of Theorem 1.4. As a result, we obtain a recursive method of generating the eigenvector matrix. Moreover, we find a nice explanation for the eigenvalue behavior. Our methods yield the following generalization: if any matrix is generated in the same way as the $A_n$, then it must be essentially binomial.

***Definition 5.1:*** Let $B$ be a 2-by-2 matrix, $B = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. Define an $n$-by-$n$ matrix $S_n(B)$ as follows: the $j^{\text{th}}$ entry of the $i^{\text{th}}$ row of $S_n(B)$ is given by

$$S_n(B)(i, j) = \text{the coefficient of } x^{n-j}y^{j-1} \text{ in } (ax+by)^{n-i}(cx+dy)^{i-1}.$$

Then, for $A = A_2 = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, by the binomial theorem we have $A_n = S_n(A)$. For general $B$ as in Definition 5.1, we let $B_n = S_n(B)$. Thus, we have

$$B_2 = B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad B_3 = \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{pmatrix},$$

$$B_4 = \begin{pmatrix} a^3 & 3a^2b & 3ab^2 & b^3 \\ a^2c & 2abc+a^2d & 2abd+b^2c & b^2d \\ ac^2 & 2acd+bc^2 & 2bcd+ad^2 & bd^2 \\ c^3 & 3c^2d & 3cd^2 & d^3 \end{pmatrix}.$$

***Lemma 5.2:*** Let $B = B_2$ be a 2-by-2 matrix, $B = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and let $B_n = S_n(B)$. Then $B_n$ may be generated recursively from $B_{n-1}$ by

$$B_n(i, j) = bB_{n-1}(i, j-1) + aB_{n-1}(i, j),$$
$$B_n(n, j) = \binom{n-1}{j-1} c^{n-j} d^{j-1},$$

with the convention that $B_n(i, j) = 0$ for $j < 1$ or $j > n$.

***Proof:*** Induction on $n$. □

In order to prove the next lemmas, we need to define some notation.

***Definition 5.3:*** Let $R = \mathbb{C}[x, y]$ be the ring over $\mathbb{C}$ in two indeterminates. Define $V_n$ to be the $\mathbb{C}$-vector space of homogeneous polynomials in $R$ of degree $n-1$. A basis for $V_n$ is $\{x^{n-1}, x^{n-2}y, \dots, y^{n-1}\}$, so $V_n$ is of dimension $n$. Any 2-by-2 matrix $B = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ induces a ring homomorphism, $\phi_B : R \to R$, by sending $x$ to $ax+by$ and sending $y$ to $cx+dy$. Since $\phi_B$ is degree-preserving and linear in $x$ and $y$, it induces a linear transformation on $V_n$. We denote this linear transformation by $\phi_{B,n}$.

***Lemma 5.4:*** If we write an element $V_n$ as a row vector with respect to the basis $\{x^{n-1}, x^{n-2}y, \ldots, y^{n-1}\}$, then the action of $\phi_{B,n}$ on $V_n$ is given by multiplication by $S_n(B)$ on the right.

***Proof:*** The $i^{\text{th}}$ basis vector of $V_n$, namely, the vector $x^{n-i}y^{i-1}$, goes to $(ax+by)^{n-i}(cx+dy)^{i-1}$ under the linear transformation $\phi_{B,n}$, which just forms the $i^{\text{th}}$ row of $S_n(B)$. □

***Lemma 5.5:*** Let $B$ and $C$ be 2-by-2 matrices. Then $S_n(BC) = S_n(B)S_n(C)$.

***Proof:*** The matrix equation $BC = B \cdot C$ gives rise to the ring homomorphism equation $\phi_{BC} = \phi_C \circ \phi_B$ (note that the matrices act on the right; hence, $\phi_B$ is applied first). Since the ring homomorphisms act the same on $V_n$, we obtain the equality of linear transformations $\phi_{BC,n} = \phi_{C,n}\phi_{B,n}$. Now, by Lemma 5.4, we obtain $S_n(BC) = S_n(B)S_n(C)$ □

***Theorem 5.6:*** Let $G_n = S_n(E_2)$, where $E_2 = \begin{bmatrix} \gamma & -1/\gamma \\ 1 & 1 \end{bmatrix}$ is the eigenvector matrix for $A_2$. Then

$$A_n G_n = G_n D_n.$$

Hence, $G_n$ is the eigenvector matrix for $A_n$ (scaled so that the bottom row of $G_n$ is the top row of $A_n$) and $D_n$ is the diagonalization of $A_n$, giving the eigenvalues of $A_n$ to be $(-1)^{n-i}\gamma^{2i-n+1}$ as $i$ ranges from 0 to $n-1$.

***Proof:*** As we have observed after Definition 5.1, we have

$$S_n(A_2) = A_n. \tag{5.7}$$

If we start with the matrix equation $A_2E_2 = E_2D_2$ and apply the operator $S_n$, then, from Lemma 5.5 and equation (5.7), we obtain

$$A_nS_n(E_2) = S_n(E_2)S_n(D_2).$$

The action of $D_2$ on $V_2$ sends $x$ to $\gamma x$ and $y$ to $-\gamma^{-1}y$. Therefore, $S_n(D_2)$ sends $x^iy^{n-1-i}$ to $(-1)^{n-1-i}\gamma^{2i-n+1}x^iy^{n-1-i}$, which is exactly the action of $D_n$. Thus, $D_n = S_n(D_2)$ and, consequently, $S_n(E_2)$ must be the eigenvector matrix. This gives the result. □

***Remark:*** These results can be interpreted in terms of the symmetric algebra of $\mathbb{C}^2$, denoted Sym $\mathbb{C}^2$ (see [2], p. 141). If $e_1$ and $e_2$ are a basis for $\mathbb{C}^2$, the ring $R$ above is isomorphic to the symmetric algebra of $\mathbb{C}^2$ by sending $x$ to $e_1$ and $y$ to $e_2$. The set of homogeneous polynomials of degree $n$ of $R$ is just the $(n-1)$-fold symmetric tensor product of $\mathbb{C}^2$, denoted $\text{Sym}^{n-1}\mathbb{C}^2$. As we have observed above, the linear transform $A_2$ acting on $\mathbb{C}^2$ induces an action on Sym $\mathbb{C}^2$.

Lemma 5.2 gives an explicit means for computing the eigenvector matrix. Since $G_n = S_n(E_2)$, we have the following recursive method for computing the eigenvector matrix.

***Corollary 5.8:***

$$G_n(i,j) = -1/\gamma G_{n-1}(i,j-1) + \gamma G_{n-1}(i,j),$$
$$G_n(n,j) = \binom{n-1}{j-1}.$$

Similarly, we can compute the inverse of the eigenvalue matrix so that the explicit diagonalization of $A_n$ can be given. We have

$$H_2 = G_2^{-1} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 1/\gamma \\ -1 & \gamma \end{bmatrix}.$$

Thus, since $H_n = G_n^{-1} = S_n(G_2^{-1})$, we obtain the following result from Lemma 5.2.

***Corollary 5.9:***

$$H_n(i, j) = \frac{1}{\sqrt{5}}[1/\gamma B_{n-1}(i, j-1) + B_{n-1}(i, j)],$$

$$H_n(n, j) = \sqrt{5}^{n-1}\binom{n-1}{j-1}(-1)^{n-j}\gamma^{j-1}.$$

We note that the proof of Theorem 5.6 actually shows the following generalization.

***Theorem 5.10:*** Let $B$ be a 2-by-2 matrix with distinct nonzero eigenvalues $\alpha$ and $\alpha'$. Then eigenvalues of $S_n(B)$ are $\alpha^{(n-i)}\alpha'^{(i-1)}$, where $i$ ranges from 1 to $n$. Moreover, if $E$ is the eigenvector matrix for $B$, then the eigenvector matrix for $S_n(B)$ is $S_n(E)$.

However, we also note that if the set of matrices $S_n(B)$ comes from a single array of numbers as the inverted binomial matrices (the $A_n$) do, then the array of numbers must be essentially binomial.

***Theorem 5.11:*** Let $B$ be a 2-by-2 matrix. Suppose that the entries of the matrix $S_n(B)$ come from a single array of numbers for each $n > 1$. Then $B$ must be of the form $\left[\begin{smallmatrix} a & b \\ 1 & 0 \end{smallmatrix}\right]$. In this case, the entries of the $i^{\text{th}}$ row of $S_n(B)$ are just the coefficients of $(ax+by)^{n-i}$.

***Proof:*** Assume $B = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. Then

$$S_3(B) = \begin{bmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{bmatrix},$$

so that we must have $c^2 = c, 2cd = d, ac = a, ad+bc = b$. These imply that $c = 1$ and $d = 0$. Then the entries of the $i^{\text{th}}$ row of $S_n(B)$ are the coefficients of $(ax+by)^{n-i}$ and, hence, just the binomial matrix scaled by powers of $a$ and $b$. □

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