THE BRAHMAGUPTA POLYNOMIALS

E. R. Suryanarayan
Department of Mathematics, University of Rhode Island, Kingston, RI 02881
(*Submitted May 1994*)

1. INTRODUCTION

In this paper we define the Brahmagupta matrix [see (1), below] and show that it generates a class of homogeneous polynomials x_n and y_n in x and y satisfying a host of relations; the polynomials contain as special cases the well-known Fibonacci, Lucas, and Pell sequences, and the sequences observed by Entringer and Slater [1], while they were investigating the problem of information dissemination through telegraphs; x_n and y_n also include the Fibonacci polynomials, the Pell and Pell-Lucas polynomials [5], [7], and the Morgan-Voyce polynomials in Ladder Networks and in Electric Line Theory [6], [9]. We also extend some series and convolution properties that hold for the Fibonacci and Lucas sequences and the Pell and Pell-Lucas polynomials to x_n and y_n [7], [5].

2. THE BRAHMAGUPTA MATRIX

To solve the indeterminate equation $x^2 = ty^2 \pm m$ in integers, where t is square free, the Indian astronomer and mathematician Brahmagupta (*ca.* 598) gave an iterative method of deriving new solutions from the known ones by his *samasa-bhavana*, the principle of composition: If (x_1, y_1, m_1) and (x_2, y_2, m_2) are trial solutions of the indeterminate equation, then the triple $(x_1x_2 \pm ty_1y_2, x_1y_2 \pm y_1x_2, m_1m_2)$ is also a solution of the indeterminate equation which can be expressed using the multiplication rule for a 2 by 2 matrix. Notice that if we set

$$B(x, y) = \begin{bmatrix} x & y \\ \pm ty & \pm x \end{bmatrix}, \tag{1}$$

and $m = \det B$, then the results

$$B(x_1, y_1)B(x_2, y_2) = B(x_1x_2 \pm ty_1y_2, x_1y_2 \pm y_1x_2), \tag{2}$$

$$\det[B(x_1, y_1)B(x_2, y_2)] = \det B(x_1, y_1)\det B(x_2, y_2) = m_1m_2, \tag{3}$$

give the Brahmagupta rule. Equation (3) is usually referred to as the *Brahmagupta Identity* and appears often in the history of number theory [10].

Let **M** denote the set of matrices of the form

$$B = \begin{bmatrix} x & y \\ ty & x \end{bmatrix}, \tag{4}$$

where t is a fixed real number and x and y are variables. Define B to be the Brahmagupta matrix. **M** satisfies the following properties:

1. **M** is a field for $x, y, t \in R$ and $t < 0$; in particular, if $t = -1$, then we have the well-known one-to-one correspondence between the set of matrices and the complex numbers $x + iy$:

$$\begin{bmatrix} x & y \\ -y & x \end{bmatrix} \leftrightarrow x + iy.$$

2. The following eigenrelations, in which T denotes the transpose, hold:
$$B[1, \pm\sqrt{t}]^T = (x \pm y\sqrt{t})[1, \pm\sqrt{t}]^T,$$
and these relations imply
$$B^n[1, \pm\sqrt{t}]^T = (x \pm y\sqrt{t})^n[1, \pm\sqrt{t}]^T$$
and
$$\begin{bmatrix} x & y \\ ty & x \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \\ \sqrt{\frac{t}{2}} & -\sqrt{\frac{t}{2}} \end{bmatrix} \begin{bmatrix} x + y\sqrt{t} & 0 \\ 0 & x - y\sqrt{t} \end{bmatrix} \begin{bmatrix} \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2t}} \\ \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2t}} \end{bmatrix}.$$
Define
$$B^n = \begin{bmatrix} x & y \\ ty & x \end{bmatrix}^n = \begin{bmatrix} x_n & y_n \\ ty_n & x_n \end{bmatrix} = B_n.$$
3. Then the following recurrence relations are satisfied:
$$x_{n+1} = xx_n + tyy_n,\ y_{n+1} = xy_n + yx_n, \tag{5}$$
with $x_n = x$ and $y_n = y$.
4. Using the above eigenrelations, we derive the following Binet forms for x_n and y_n:
$$x_n = \frac{1}{2}[(x + y\sqrt{t})^n + (x - y\sqrt{t})^n], \tag{6}$$
$$y_n = \frac{1}{2\sqrt{t}}[(x + y\sqrt{t})^n - (x - y\sqrt{t})^n], \tag{7}$$
and $x_n \pm \sqrt{t}y_n = (x \pm \sqrt{t}y)^n$.
5. Let $\xi_n = x_n + y_n\sqrt{t}$, $\eta_n - y_n\sqrt{t}$, and $\beta_n = x_n^2 - ty_n^2$, with $\eta_n = \eta$, $\xi_n = \xi$, and $\beta_n = \beta$; we then have $\xi_n = \xi^n$, $\eta_n = \eta^n$, and $\beta_n = \beta^n$. To show the last equality, consider $\beta^n = (x^2 - ty^2)^n = \xi^n\eta^n = \xi_n\eta_n = (x_n^2 - ty_n^2) = \beta_n$. Notice that $\beta = \det B$.
6. The recurrence relations (5) also imply that x_n and y_n satisfy the difference equations:
$$x_{n+1} = 2x\,x_n - \beta x_{n-1},\ y_{n+1} = 2x\,y_n - \beta y_{n-1}. \tag{8}$$
Conversely, if $x_0 = 1$, $x_1 = x$, and $y_0 = 0$, $y_1 = y$, then the solutions of the difference equations (8) are indeed given by the Binet forms (6) and (7).
7. Notice that if we set $x = 1/2 = y$ and $t = 5$, then $\beta = -1$ and $2y_n = F_n$ is the Fibonacci sequence, while $2x_n = L_n$ is the Lucas sequence, where $n > 0$. For the number-theoretic properties of F_n and L_n, the reader is referred to [2] and [3].
8. In particular, if $x = y = 1$ and $t = 2$, then both x_n and y_n satisfy $x_n^2 - 2y_n^2 = (-1)^n$ and they generate the Pell sequences:
$$x_n = 1, 1, 3, 7, 17, 41, 99, 239, 577, \ldots,\ y_n = 0, 1, 2, 5, 12, 29, 70, 169, 408, \ldots.$$
It is interesting to note that if we set
$$a = 2(x_n + y_n)y_n\ \ b = x_n(x_n + 2y_n),\ c = x_n^2 + 2x_ny_n + 2y_n^2,$$

we then obtain integral solutions of the Pythagorean relation $a^2+b^2=c^2$, where a and b are consecutive integers [8].

9. If $t=1$, then $x_n+y_n=(x+y)^n$, and if $t=-1$, then $x_n+iy_n=(x+iy)^n$. Also, for every square free integer t, the set of matrices $\mathbf{M}$ is isomorphic to the set $\{x+y\sqrt{t}\,|\,x,y\in Z\}$, where Z is the set of integers.

10.

$$e^B=\frac{1}{4}\begin{bmatrix} e^{\xi}+e^{\eta} & \frac{1}{\sqrt{t}}(e^{\xi}-e^{\eta}) \\ \sqrt{t}(e^{\xi}-e^{\eta}) & e^{\xi}+e^{\eta}\end{bmatrix},\quad \det e^B=e^{2x}.$$

To show these results, let us write $2x_k=\xi^k+\eta^k$, $2\sqrt{t}\,y_k=\xi^k-\eta^k$. Since

$$e^B=\sum_{k=0}^{\infty}\frac{B^k}{k!}\quad\text{and}\quad \frac{B^k}{k!}=\frac{1}{k!}\begin{bmatrix} x_k & y_k \\ ty_k & x_k\end{bmatrix},$$

we express x_k and y_k in terms of ξ and η and obtain the desired results.

11. x_n and y_n can be extended to negative integers by defining $x_{-n}=x_n\beta^{-n}$ and $y_{-n}=-y_n\beta^{-n}$. We will then have

$$B^{-n}=\begin{bmatrix} x & y \\ ty & x\end{bmatrix}^{-n}=\begin{bmatrix} x_{-n} & y_{-n} \\ ty_{-n} & x_{-n}\end{bmatrix}=B_{-n};$$

here we have used the property

$$\left(\begin{bmatrix} x & y \\ ty & x\end{bmatrix}^{-1}\right)^n=\left(\frac{1}{\beta}\begin{bmatrix} x & -y \\ -ty & x\end{bmatrix}\right)^n=\frac{1}{\beta^n}\begin{bmatrix} x_n & -y_n \\ -ty_n & x_n\end{bmatrix}.$$

All the recurrence relations extend to the negative integers, also. Notice that $B^0=I$, the identity matrix.

3. THE BRAHMAGUPTA POLYNOMIALS

1. Using the Binet forms (6) and (7), we can deduce a number of results. Write x_n and y_n as polynomials in x and y using the binomial expansion:

$$x_n=x^n+t\binom{n}{2}x^{n-2}y^2+t^2\binom{n}{4}x^{n-4}y^4+\cdots,$$

$$y_n=nx^{n-1}y+t\binom{n}{3}x^{n-3}y^3+t^2\binom{n}{5}x^{n-5}y^5+\cdots.$$

Notice that x_n and y_n are homogeneous in x and y. The first few polynomials are

$$x_0=1,\ x_1=x,\ x_2=x^2+ty^2,\ x_3=x^3+3tyx^2,\ x_4=x^4+6tx^2y^2+t^2y^4,\ldots,$$

$$y_0=0,\ y_1=y,\ y_2=2xy,\ y_3=3x^2y+ty^3,\ y_4=4x^3y+4txy^3,\ldots.$$

2. If $t>0$, then x_n and y_n satisfy

$$\lim_{n\to\infty}\frac{x_n}{y_n}=\sqrt{t},\quad \lim_{n\to\infty}\frac{x_n}{x_{n-1}}=\lim_{n\to\infty}\frac{y_n}{y_{n-1}}=x+\sqrt{t}\,y.$$

3.
$$\frac{\partial x_n}{\partial x}=\frac{\partial y_n}{\partial y}=nx_{n-1},$$
$$\frac{\partial x_n}{\partial y}=t\frac{\partial y_n}{\partial y}=nty_{n-1}.$$

From the above relations, we infer that x_n and y_n are the polynomial solutions of the wave equation
$$\left(\frac{\partial^2}{\partial x^2}-\frac{1}{t}\frac{\partial^2}{\partial y^2}\right)U=0.$$

4. If $\beta=-1$, then $ty^2=x^2+1$, then the difference equations (8) become
$$x_{n+1}=2x\,x_n+x_{n-1},\quad y_{n+1}=2xy_n+y_{n-1}. \tag{9}$$

5. If $2x=\alpha$ and $\beta=1$, $x_1=1$, $x_2=\alpha$, and $y_1=1$, $y_2=\alpha-1$, then x_n and y_n generate Morgan-Voyce polynomials [6], [9].

4. RECURRENCE RELATIONS

1. From the Binet forms (6) and (7), we can derive the following recurrence relations:

$$\begin{array}{rl}
\text{(i)} & x_{m+n}=x_mx_n+ty_my_n,\\
\text{(ii)} & y_{m+n}=x_my_n+y_mx_n,\\
\text{(iii)} & \beta^n x_{m-n}=x_mx_n-ty_my_n,\\
\text{(iv)} & \beta^n y_{m-n}=x_ny_m-x_my_n,\\
\text{(v)} & x_{m+n}+\beta^n x_{m-n}=2x_mx_n,\\
\text{(vi)} & y_{m+n}+\beta^n y_{m-n}=2x_ny_m,\\
\text{(vii)} & x_{m+n}-\beta^n x_{m-n}=2ty_my_n,\\
\text{(viii)} & y_{m+n}-\beta^n y_{m-n}=2x_my_n,\\
\text{(ix)} & 2(x_m^2-x_{m+n}x_{m-n})=\beta^{m-n}(\beta^n-x_{2n}),\\
\text{(x)} & x_{2m}-2ty_{m+n}y_{m-n}=\beta^{m-n}x_{2n}.
\end{array} \tag{10}$$

2. Put $m=n$ in (i) and (ii) above; then we see that
$$x_{2n}=x_n^2+ty_n^2,\quad y_{2n}=2x_ny_n,$$
and these relationships imply that
(a) x_{2n} is divisible by $x_n\pm i\sqrt{t}y_n$, if $t>0$,
(b) x_{2n} is divisible by $x_n\pm i\sqrt{t}y_n$, if $t<0$,
(c) y_{2n} is divisible by x_n and y_n; also, if r divides s, then x_{rn} and y_{rn} are divisors of y_{sn}.

3. Let $\Sigma_{k=1}^n=\Sigma$. Then, using the Binet forms, we can also derive the following relations:

$$
\begin{aligned}
&\text{(i)} && \sum x_k = \frac{\beta x_n - x_{n+1} + x - \beta}{\beta - 2x + 1}, \\
&\text{(ii)} && \sum y_k = \frac{\beta y_n - y_{n+1} + y}{\beta - 2x + 1}, \\
&\text{(iii)} && \sum x_k^2 = \frac{\beta^2 x_{2n} - x_{2n+2} + x_2 - \beta^2}{2(\beta^2 - 2x_2 + 1)} + \frac{\beta(\beta^n - 1)}{2(\beta - 1)}, \\
&\text{(iv)} && \sum y_k^2 = \frac{\beta^2 x_{2n} - x_{2n+2} + x_2 - \beta^2}{2t(\beta^2 - 2x_2 + 1)} - \frac{\beta(\beta^n - 1)}{2t(\beta - 1)}, \\
&\text{(v)} && 2\sum x_k x_{n+1-k} = nx_{n+1} + \frac{\beta y_n}{y}, \\
&\text{(vi)} && 2t\sum y_k y_{n+1-k} = nx_{n+1} - \frac{\beta y_n}{y}, \\
&\text{(vii)} && 2\sum x_k y_{n-k+1} = 2\sum y_k x_{n-k+1} = ny_{n+1}.
\end{aligned}
\tag{11}
$$

4. Now we show an interesting result which generalizes a property that holds between F_n and L_n, namely, $e^{L(x)} = F(x)$, where

$$F(x) = F_1 + F_2 x + F_3 x^2 + \cdots + F_{n+1} x^n + \cdots,$$

and

$$L(x) = L_1 x + \frac{L_2}{2}x^2 + \frac{L_3}{3}x^2 + \cdots + \frac{L_n}{n}x^n + \cdots$$

(see [4]). Let X and Y be generating functions of x_n and y_n, respectively; that is,

$$X = \sum_1^\infty \frac{x_n}{n}s^n, \quad Y = \sum_1^\infty y_n s^n,$$

then $Y(s) = sye^{2X(s)}$. To prove this result, consider $Y(s) = y_1 s + y_2 s^2 + y_3 s^3 + \cdots + y_n s^n + \cdots$. Then $sY(s) = y_1 s^2 + y_2 s^3 + y_3 s^4 + \cdots + y_n s^{n+1} + \cdots$, and $s^2 Y(s) = y_1 s^3 + y_2 s^4 + \cdots + y_n s^{n+2} + \cdots$. Substituting the power series for $Y(s)$ into the expression $Y(x) - 2xsY(s) + \beta s^2 Y(s)$, we obtain

$$[1 - 2xs + \beta s^2]Y(s) = ys + \sum_{k=1}^\infty [y_{k+1} - 2xy_k + \beta y_{k-1}]s^{k+1},$$

where we have put $y_1 = y$. Now, using the property $y_{k+1} - 2xy_k + \beta y_{k-1} = 0$ in equation (8), we find that the above expression reduces to

$$[1 - 2xs + \beta s^2]Y(s) = ys. \tag{12}$$

Now consider the series

$$X(s) = x_1 s + \frac{x_2}{2}s^2 + \frac{x_3}{3}s^3 + \cdots + \frac{x_n}{n}s^n + \cdots,$$

and in it express x_n in terms of ξ_n and η_n to get

$$X(s) = \frac{1}{2}(\zeta + \eta)s + \frac{1}{2}\left[\frac{1}{2}(\xi^2 + \eta^2)\right]s^2 + \frac{1}{3}\left[\frac{1}{2}(\xi^3 + \eta^3)\right]s^3 + \cdots + \frac{1}{n}\left[\frac{1}{2}(\xi^n + \eta^n)\right]s^n + \cdots,$$

which can be rewritten in the form

$$X(s)=\frac{1}{2}\left[\xi s+\frac{1}{2}\xi^2s^2+\frac{1}{3}\xi^3s^3+\cdots\right]+\frac{1}{2}\left[\eta s+\frac{1}{2}\eta^2s^2+\frac{1}{3}\eta^3s^3+\cdots\right].$$

Therefore,

$$X(s)=-\frac{1}{2}\ln(1-\xi s)-\frac{1}{2}\ln(1-\eta s)=-\frac{1}{2}\ln[(1-\xi s)(1-\eta s)].$$

Since $(1-\xi s)(1-\eta s)=1-2xs+\beta s^2$, we have

$$2X(s)=-\ln[1-2xs+\beta s^2]. \tag{13}$$

Now compare (12) and (13) and obtain the desired result: $Y(s)=sye^{2X(s)}$.

5. SERIES SUMMATION INVOLVING RECIPROCALS OF x_n AND y_n

Let us look at some infinite series summations involving x_n and y_n and extend some of the infinite series results that are known for F_n and L_n [4] and for $P_n(x)$ and $Q_n(x)$ [7] to x_n and y_n. First, we shall show that

1. $$\sum_{k=1}^{\infty}\frac{1}{x_{k+1}}\left(\frac{2x}{x_{k-1}}-\frac{\beta+1}{x_k}\right)=\frac{1}{x}.$$

To show the above result, consider

$$\frac{1}{x_{k-1}x_k}-\frac{1}{x_kx_{k+1}}=\frac{x_{k+1}-x_{k-1}}{x_{k-1}x_kx_{k+1}}$$
$$=\frac{2xx_k-\beta x_{k-1}-x_{k-1}}{x_{k-1}x_kx_{k+1}}=\frac{2x}{x_{k-1}x_{k+1}}-\frac{\beta+1}{x_kx_{k+1}},$$

where we have used the property $x_{k+1}=2xx_k-\beta x_{k-1}$. Therefore,

$$\sum_{k=1}^{\infty}\frac{1}{x_{k+1}}\left(\frac{2x}{x_{k-1}}-\frac{\beta+1}{x_k}\right)=\sum_{k=1}^{\infty}\left(\frac{1}{x_{k-1}x_k}-\frac{1}{x_kx_{k+1}}\right)=\frac{1}{x_0x_1}=\frac{1}{x}.$$

In particular, if $ty^2=1+x^2$ and $y=1$, then $\beta=-1$, and the above result reduces to

$$\sum_{k=1}^{\infty}\frac{1}{x_{k-1}x_{k+1}}=\frac{1}{2x^2},$$

where x_k is given by equation (9). Similarly, we can show that

2. $$\sum_{k=r+1}^{\infty}\left(\frac{2x}{x_{k-1}x_{k+1}}-\frac{\beta+1}{x_{k+1}x_k}\right)=\frac{1}{x_rx_{r+1}}.$$

For the special case $ty^2=1+x^2$ and $y=1$, then $\beta=-1$ and the above result becomes

$$\sum_{k=r+1}^{\infty}\frac{1}{x_{k-1}x_{k+1}}=\frac{1}{2xx_rx_{r+1}},$$

where x_k is given by equation (9). Following a similar argument, we can show

3.
$$\sum_{k=r+1}^{\infty}\left(\frac{2x}{y_{k-1}y_{k+1}}-\frac{\beta+1}{y_k y_{k+1}}\right)=\frac{1}{y_r y_{r+1}}.$$

Again using $2xx_k = x_{k+1}+\beta x_{k-1}$, we can derive

4.
$$\sum_{k=r+1}^{\infty}\frac{2xx_k}{x_{k-1}x_{k+1}}=\sum_{k=r+1}^{\infty}\left(\frac{1}{x_{k-1}}+\frac{\beta}{x_{k+1}}\right).$$

If $ty^2 = 1+x^2$ and $y=1$, then we have

$$\sum_{k=r+1}^{\infty}\frac{x_k}{x_{k-1}x_{k+1}}=\frac{1}{2x}\left(\frac{1}{x_r}+\frac{1}{x_{r+1}}\right),$$

where x_k is given by equation (9). Similarly, from the recurrence relation $2xy_k = y_{k+1}+\beta y_{k-1}$, we have

5.
$$\sum_{k=r+1}^{\infty}\frac{2xy_k}{y_{k-1}y_{k+1}}=\sum_{k=r+1}^{\infty}\left(\frac{1}{y_{k-1}}+\frac{\beta}{y_{k+1}}\right).$$

In particular, if $ty^2 = 1+x^2$ and $y=1$, then $\beta=-1$ and the above result becomes

$$\sum_{k=r+1}^{\infty}\frac{y_k}{y_{k-1}y_{k+1}}=\frac{1}{2x}\left(\frac{1}{y_r}+\frac{1}{y_{r+1}}\right),$$

where y_k is now given by equation (9).

6. Now we generalize the results of items 2 and 3 of this section; we shall show that

$$\sum_{k=2}^{\infty}\frac{1}{x_{(k+1)r}}\left(\frac{2x_r}{x_{(k-1)r}}-\frac{\beta^r+1}{x_{kr}}\right)=\frac{1}{x_r x_{2r}}.$$

To show this, we consider the left-hand side of the above result:

$$\sum_{k=2}^{\infty}\frac{1}{x_{(k+1)r}}\left(\frac{2x_r x_{kr}-\beta^r x_{(k-1)r}-x_{(k-1)r}}{x_{(k-1)r}x_{kr}}\right).$$

The above result can be simplified by using property (v) of (10), with $m=rk$, $n=r$, that is, $x_{rk+r}+\beta^r x_{rk-r}=2x_{rk}x_r$. Then the above expression becomes

$$\sum_{k=2}^{\infty}\frac{1}{x_{(k+1)r}}\left(\frac{x_{(k+1)r}-x_{(k-1)r}}{x_{(k-1)r}x_{kr}}\right),$$

which reduces to

$$\sum_{k=2}^{\infty}\left(\frac{1}{x_{(k-1)r}x_{kr}}-\frac{1}{x_{kr}x_{(k+1)r}}\right),$$

which, when summed over k, reduces to $1/(x_r x_{2r})$. Similarly, we show that

$$\sum_{k=2}^{\infty}\frac{1}{y_{(k+1)r}}\left(\frac{2x_r}{y_{(k-1)r}}-\frac{\beta^r+1}{y_{kr}}\right)=\frac{1}{y_r y_{2r}}.$$

7. Let us now generalize a property that holds for Fibonacci series [4]. For $t > 0$, consider the series:

$$S = \sum_{k=2}^{\infty} \frac{\beta^{2^{k-1}-2} y_2}{y_{2^k}} = \frac{y_2}{y_4} + \frac{\beta^2 y_2}{y_8} + \frac{\beta^6 y_2}{y_{16}} + \cdots.$$

Denote

$$S_n = \frac{y_2}{y_4} + \frac{\beta^2 y_2}{y_8} + \frac{\beta^6 y_2}{y_{16}} + \cdots + \frac{\beta^{2^{n-1}-2} y_2}{y_{2^n}}.$$

By induction we shall show that

$$S_n = \frac{y_{2^n-2}}{y_{2^n}}. \tag{14}$$

Note that $y_{2^n} = 0$ implies that either $x = 0$ or $y = 0$ from the Binet form for y_{2^n}. Therefore, we shall assume that $y_{2^n} \neq 0$. Observe also that equation (12) is true for $n = 2, 3$. Consider

$$S_{n+1} = S_n + \frac{\beta^{2^n-2} y_2}{y_{2^{n+1}}} = \frac{y_{2^n-2}}{y_{2^n}} + \frac{\beta^{2^n-2} y_2}{y_{2^{n+1}}} = \frac{y_{2^{n+1}} y_{2^n-2} + y_{2^n} \beta^{2^n-2} y_2}{y_{2^n} y_{2^{n+1}}}.$$

Use the property $y_{2m} = 2x_m y_m$, with $m = 2^{n+1} = 2(2^n)$, to get

$$S_{n+1} = \frac{2x_{2^n} y_{2^n} y_{2^n-2} + y_{2^n} \beta^{2^n-2} y_2}{y_{2^n} y_{2^{n+1}}} = \frac{2x_{2^n} y_{2^n-2} + \beta^{2^n-2} y_2}{y_{2^{n+1}}}.$$

Now recall property (viii) of equation (10), $y_{m+p} - \beta^p y_{m-p} = 2x_m y_p$, and in it set $m = 2^n$, $p = 2^n - 2$. We then have

$$S_{n+1} = \frac{y_{2^{n+1}-2}}{y_{2^{n+1}}},$$

which completes the induction. Therefore, for $t > 0$, we have

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{y_{2^n-2}}{y_{2^n}} = \xi^{-2} = \frac{1}{(x + y\sqrt{t})^2}.$$

6. CONVOLUTIONS FOR x_n AND y_n

Given two homogeneous polynomial sequences $a_n(x, y)$ and $b_n(x, y)$ in two variables x and y, where n is an integer ≥ 1, their *first convolution sequence* is defined by

$$(a_n * b_n)^{(1)} = \sum_{j=1}^{n} a_j b_{n+1-j} = \sum_{j=1}^{n} b_j a_{n+1-j}.$$

In the above definition, we have written $a_n = a_n(x, y)$ and $b_n = b_n(x, y)$. Denote $x_n * x_n = X_n^{(1)}$, $y_n * y_n = Y_n^{(1)}$, $2x_n * y_n = y_n^{(1)}$, and $X_n^{(1)} + tY_n^{(1)} = x_n^{(1)}$. To determine these convolutions, we use the matrix properties of B, namely,

$$\begin{bmatrix} x & y \\ ty & x \end{bmatrix}^{n+1} = \begin{bmatrix} x_{n+1} & y_{n+1} \\ ty_{n+1} & x_{n+1} \end{bmatrix} = B^{n+1} = B^j B^{n+1-j} = \begin{bmatrix} x_j & y_j \\ ty_j & x_j \end{bmatrix} \begin{bmatrix} x_{n+1-j} & y_{n+1-j} \\ ty_{n+1-j} & x_{n+1-j} \end{bmatrix}.$$

Let

$$B_n^{(1)} = \sum_{j=1}^{n} B_j B_{n+1-j} = \sum_{j=1}^{n} B^{n+1}.$$

Note that $B^n = B_n$. We prefer using the subscript notation. Since $\sum_{j=1}^{n} B_{n+1} = nB_{n+1}$, we have

$$nB_{n+1} = \sum_{j=1}^{n} \begin{bmatrix} x_j & y_j \\ ty_j & x_j \end{bmatrix} \begin{bmatrix} x_{n+1-j} & y_{n+1-j} \\ ty_{n+1-j} & x_{n+1-j} \end{bmatrix}.$$

Let $\sum_{j=1}^{n} = \sum$, then the above result can be written

$$nB_{n+1} = \begin{bmatrix} \sum x_j x_{n+1-j} + t\sum y_j y_{n+1-j} & \sum x_j y_{n+1-j} + \sum y_j x_{n+1-j} \\ t\left(\sum x_j y_{n+1-j} + \sum y_j x_{n+1-j}\right) & \sum x_j x_{n-j} + t\sum y_j y_{n+1-j} \end{bmatrix},$$

or

$$nB_{n+1} = \begin{bmatrix} x_n * x_n + ty_n * y_n & 2x_n * y_n \\ 2tx_n * y_n & x_n * x_n + ty_n * y_n \end{bmatrix} = \begin{bmatrix} x_n^{(1)} & y_n^{(1)} \\ ty_n^{(1)} & x_n^{(1)} \end{bmatrix} = B_n^{(1)}.$$

Therefore, we have

$$x_n^{(1)} = nx_{n+1}, \quad y_n^{(1)} = ny_{n+1}.$$

The above result can be extended to the k^{th} convolution by defining

$$B_n^{(k)} = \sum_{j=1}^{n} B_j (B^{(k-1)})_{n+1-j}.$$

Now we shall show that

$$B_n^{(k)} = \binom{n+k-1}{k} B_{n+k}.$$

We shall prove the result by induction on k. Since $B^{(1)} = nB_{n+1}$, the result is true for $k = 1$. Now consider

$$B_n^{(k+1)} = \sum B_j B_{n+1-j}^{(k)} = \sum B_{n+1-j} (B^{(k)})_j$$
$$= \sum B_{n+1-j} \binom{j+k-1}{k} B_{j+k} = B_{n+k+1} \sum \binom{j+k-1}{k} = \binom{n+k}{k+1} B_{n+k+1},$$

which completes the induction.

From the above results, we write the k^{th} convolution of x_n and y_n:

$$x_n^{(k)} = \binom{n+k-1}{k} x_{n+k}, \quad y_n^{(k)} = \binom{n+k-1}{k} y_{n+k}. \tag{15}$$

Also, from properties (v) and (vi) of (10), we have

$$2X^{(1)} = nx_{n+1} + \frac{\beta y_n}{y}, \quad 2tY^{(1)} = nx_{n+1} - \frac{\beta y_n}{y}, \tag{16}$$

which can be written in the form

$$2X^{(1)} = x_n^{(1)} + \frac{\beta y_n}{y}, \quad 2tY^{(1)} = x_n^{(1)} - \frac{\beta y_n}{y}.$$

We can also extend the above result to the k^{th} convolution of x_n and y_n, namely,

$$x_n * x_n^{(k)},\ y_n * y_n^{(k)},\ x_n * y_n^{(k)},\ y_n * x_n^{(k)}.$$

Using the results (10)(v), (15), and (16), and some computation, we obtain

$$2x_n * x_n^{(k)} = \binom{n+k}{k+1} x_{n+k+1} + \sum_{j=1}^{n} \binom{j+k-1}{k} \beta^{j+k} x_{n+1-2j-k}.$$

Similarly, we have

$$2ty_n * y_n^{(k)} = \binom{n+k}{k+1} x_{n+k+1} - \Sigma \binom{j+k-1}{k} \beta^{j+k} x_{n+1-2j-k},$$

$$2x_n^{(k)} * y_n = \binom{n+k}{k+1} y_{n+k+1} + \Sigma \binom{j+k-1}{k} \beta^{j+k} y_{n+1-2j-k},$$

$$2x_n * y_n^{(k)} = \binom{n+k}{k+1} y_{n+k+1} - \Sigma \binom{j+k-1}{k} \beta^{j+k} y_{n+1-2j-k}.$$

What we have seen here is but a sample of the properties displayed by the versatile matrix B. We are sure there are many more.

REFERENCES

1. R. C. Entringer & P. J. Slater. "Gossips and Telegraphs." *J. Franklin Institute* **307.6** (1979): 353.
2. G. H. Hardy and E. M. Wright. *An Introduction to Number Theory*. Oxford: Oxford University Press, 1960.
3. V. E. Hoggatt, Jr. *Fibonacci and Lucas Numbers*. Boston: Houghton Mifflin, 1969.
4. R. Honsberger. *Mathematical Gems*, pp. 102-38. New York: The Mathematical Association of America, 1985.
5. A. F. Horadam & J. M. Mahon. "Convolutions for Pell Polynomials." In *Fibonacci Numbers and Their Applications*. Dordrecht: D. Reidel, 1986.
6. J. Lahr. "Fibonacci and Lucas Numbers and the Morgan-Voyce Polynomials in Ladder Networks and in Electric Line Theory." In *Fibonacci Numbers and Their Applications.* Dordrecht: D. Reidel, 1986.
7. J. M. Mahon & A. F. Horadam. "Infinite Series Summation Involving Reciprocals of Pell Polynomials." In *Fibonacci Numbers and Their Applications*. Dordrecht: D. Reidel, 1986.
8. C. D. Olds. *Continued Fractions*, p. 118. New York: Random House and the L. W. Singer Company, 1963.
9. M. N. S. Swamy. "Further Properties of Morgan-Voyce Polynomials." *The Fibonacci Quarterly* **6.2** (1968):167-75.
10. A. Weil. *Number Theory: An Approach through History from Hammurapi to Legendre*. Boston: Birkhäuser, 1983.

AMS Classification Numbers: 01A32, 11B37, 11B39