# ADVANCED PROBLEMS AND SOLUTIONS

## *Edited by* Raymond E. Whitney

*Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.*

### PROBLEMS PROPOSED IN THIS ISSUE

**H-513** ***Proposed by Paul S. Bruckman, Salmiya, Kuwait***

Define the following quantities:

$$A=\sum_{n\geq 0}\frac{1}{(n!)^2},\quad B=\sum_{n\geq 0}\frac{1}{n!(n+1)!},\quad C=\sum_{n\geq 0}\frac{(2n)!}{(n!)^4},\quad D=\sum_{n\geq 0}\frac{(2n+2)!}{n!((n+1)!)^2(n+2)!}.$$

Prove that $A^2D=B^2C$.

**H-514** ***Proposed by Juan Pla, Paris, France***

1. Let $(L_n)$ be the generalized Lucas sequence of the recursion $U_{n+2}-2aU_{n+1}+U_n=0$ with $a$ real such that $a>1$. Prove that

$$\lim_{n\to+\infty}\frac{L_2L_{2^2}L_{2^3}\cdots L_{2^n}}{L_{2^{n+1}}}=\frac{1}{4}\frac{1}{a\sqrt{a^2-1}}.$$

2. Show that the above expression has a limit when $(L_n)$ is the *classical* Lucas sequence.

**H-515** ***Proposed by Paul S. Bruckman, Salmiya, Kuwait***

For all primes $p\neq 2,5$, let $Z(p)$ denote the entry-point of $p$ in the Fibonacci sequence. It is known that $Z(p)\,|\,(p-(\frac{5}{p}))$. Let $a(p)=(p-(\frac{5}{p}))/Z(p)$, $q=\frac{1}{2}(p-(\frac{5}{p}))$. Prove that if $p\equiv 1$ or 9 (mod 20) then

$$F_{q+1}\equiv(-1)^{\frac{1}{2}(q+a(p))}\pmod{p}. \qquad (*)$$

**H-516** ***Proposed by Paul S. Bruckman, Edmonds, WA***

Given $p$ an odd prime, let $\bar{k}(p)$ denote the **Lucas period (mod $p$)**, that is, $\bar{k}(p)$ is the smallest positive integer $m=m(p)$ such that $L_{m+n}\equiv L_n \pmod{p}$ for all integers $n$.

Prove the following:

(a) Let $u=u(p)$ denote the smallest positive integer such that $\alpha^u\equiv\beta^u\equiv 1 \pmod{p}$. Then $u=m=\bar{k}(p)$.

(b) $\bar{k}(p)$ is even for all (odd) $p$.

(c) $p \equiv 1 \pmod{\bar{k}(p)}$ iff $p = 5$ or $p \equiv \pm 1 \pmod{10}$.

(d) $p \equiv -1 + \frac{1}{2}\bar{k}(p) \pmod{\bar{k}(p)}$ iff $p = 5$ or $p \equiv \pm 3 \pmod{10}$.

**H-508** ***Proposed by H.-J. Seiffert, Berlin, Germany*** **(Corrected)**

Define the Fibonacci polynomials by $F_0(x) = 0$, $F_1(x) = 1$, $F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$, for $n \ge 2$. Show that, for all complex numbers $x$ and $y$ and all positive integers $n$,

$$F_n(x)F_n(y) = n\sum_{k=0}^{n-1} \frac{1}{k+1}\binom{n+k}{2k+1}(x+y)^k F_{k+1}\left(\frac{xy-4}{x+y}\right). \tag{1}$$

As special cases of (1), obtain the following identities:

$$F_n(x)F_n(x+1) = n\sum_{k=0}^{n-1} \frac{(-1)^{n-k+1}}{k+1}\binom{n+k}{2k+1} F_{k+1}(x^2+x+4); \tag{2}$$

$$F_n(x)F_n(4/x) = n\sum_{k=0}^{[\frac{n-1}{2}]} \frac{1}{2k+1}\binom{n+2k}{4k+1}\left(\frac{x^2+4}{x}\right)^{2k}, \quad x \ne 0; \tag{3}$$

$$F_n(x)^2 = n\sum_{k=0}^{n-1} \frac{(-1)^{n-k+1}}{k+1}\binom{n+k}{2k+1}(x^2+4)^k; \tag{4}$$

$$F_n(x)^2 = n\sum_{k=0}^{n-1} \frac{1}{k+1}\binom{n+k}{2k+1}\frac{x^{2k+2}-(-4)^{k+1}}{x^2+4}; \tag{5}$$

$$F_{2n-1}(x) = (2n-1)\sum_{k=0}^{2n-2} \frac{(-1)^k}{k+1}\binom{2n+k-1}{2k+1} x^k F_{k+1}(4/x). \tag{6}$$

## SOLUTIONS

### Recurring Theme

**H-497** ***Proposed by Mohammad K. Azarian, University of Evansville, Evansville, IN (Vol. 33, no. 2, May 1995)***

Solve the recurrence relation

$$\sum_{i=0}^{k}\left(\prod_{j=0}^{k}\frac{x_{n-j}}{x_{n-i}}\right)^r + \left(\prod_{t=0}^{k} x_{n-t}\right)^r = 0,$$

where $r$ is any nonzero real number, $n > k \ge 1$, and $x_m \ne 0$ for all $m$.

***Solution by the proposer***

First, we note that

$$\sum_{i=0}^{k}\left(\prod_{j=0}^{k}\frac{x_{n-j}}{x_{n-i}}\right)^r + \left(\prod_{t=0}^{k} x_{n-t}\right)^r = (x_n x_{n-1}\cdots x_{n-k})^r\left(\sum_{i=0}^{k}\left(\frac{1}{x_{n-i}}\right)^{r(k+1)} + 1\right) = 0.$$

Now, using the fact that $(x_n x_{n-1} \cdots x_{n-k})^r \neq 0$ and making the substitution

$$u_{n-i} = \left(\frac{1}{x_{n-i}}\right)^{r(k+1)}, \quad i = 0, 1, 2, \ldots, k,$$

we obtain the following nonhomogeneous recurrence relation of order $k$:

$$u_n + u_{n-1} + u_{n-2} + \cdots + u_{n-k} = -1. \tag{1}$$

Next, the general solution to (1) has the form $u_n = u_n^{(h)} + u_n^{(p)}$, where $u_n^{(p)}$ is a particular solution to (1) and $u_n^{(h)}$ is the general solution to the homogeneous recurrence relation

$$u_n + u_{n-1} + u_{n-2} + \cdots + u_{n-k} = 0. \tag{2}$$

We know that $u_n^{(p)}$ must be a constant $A$. Thus, substituting $A$ in (1), we obtain $u_n^{(p)} = \frac{-1}{k+1}$. To find $u_n^{(h)}$, we note that the characteristic equation associated with (2) is

$$\lambda^k + \lambda^{k-1} + \lambda^{k-2} + \cdots + \lambda + 1 = 0. \tag{3}$$

Hence, using the fact that $\lambda^{k+1} - 1 = (\lambda - 1)(\lambda^k + \lambda^{k-1} + \lambda^{k-2} + \cdots + \lambda + 1)$, the roots of (3) are $k$ distinct complex roots of unity:

$$\lambda_m = \cos\frac{2m\pi}{k+1} + i\sin\frac{2m\pi}{k+1}, \quad m = 1, 2, \ldots, k.$$

But, since $\lambda_m$ is the complex conjugate of $\lambda_{k+1-m}$ when $k$ is odd, $\lambda_{\frac{k+1}{2}} = -1$. Thus, if $k$ is odd and $k \geq 3$ $\left[\text{if } k = 1, \text{ then } u_n^{(h)} = C(-1)^n\right]$, then

$$u_n^{(h)} = C(-1)^n + \sum_{m=1}^{\frac{k-1}{2}} (A_m \cos(n\theta_m) + B_m \sin(n\theta_m)),$$

where $C$, $A_m$, and $B_m$ are constants and $\theta_m = \tan^{-1}\left(\tan\frac{2m\pi}{k+1}\right)$. If $k$ is even, then

$$u_n^{(h)} = \sum_{m=1}^{\frac{k}{2}} (A_m \cos(n\theta_m) + B_m \sin(n\theta_m)),$$

where $A_m$, $B_m$, and $\theta_m$ are as above. Therefore, the general solution to the given recurrence relation is

$$x_n = (u_n)^{-\frac{1}{r(k+1)}} = \left(u_n^{(h)} + u_n^{(p)}\right)^{-\frac{1}{r(k+1)}}.$$

***Also solved by A. Dujella and P. Bruckman.***

## Pseudo Primes

**H-498** ***Proposed by Paul S. Bruckman, Edmonds, WA***
***(Vol. 33, no. 2, May 1995)***

Let $u = u_e = L_{2^e}$, $e = 2, 3, \ldots$. Show that if $u$ is composite it is both a Fibonacci pseudoprime (FPP) and a Lucas pseudoprime (LPP). Specifically, show that $u \equiv 7 \pmod{10}$, $F_{u+1} \equiv 0 \pmod{u}$, and $L_u \equiv 1 \pmod{u}$.

***Solution by L. A. G. Dresel, Reading, England***

For convenience of writing subscripts, let $E = 2^e$ so that $u_e = L_E$ and

$$u_{e+1} = L_{2E} = (L_E)^2 - 2 = (u_e)^2 - 2. \tag{1}$$

Now make the inductive hypothesis that, for some $e \geq 2$,

$$u_e \equiv 7 \pmod{10}. \tag{2}$$

Then (1) gives $u_{e+1} = (u_e)^2 - 2 \equiv 49 - 2 \equiv 7 \pmod{10}$ and, since $u_2 = L_4 = 7$, the congruence (2) is proved for all $e \geq 2$.

Next, make the inductive hypothesis that, for some $e \geq 2$,

$$u_e = h2^{e+1} - 1, \text{ where } h \text{ is an } odd \text{ integer.} \tag{3}$$

Then

$$\begin{aligned} u_{e+1} &= (u_e)^2 - 2 = (h2^{e+1} - 1)^2 - 2 \\ &= (h^2 2^e - h)2^{e+2} - 1, \end{aligned}$$

where $(h^2 2^e - h)$ is again an odd integer, since $h$ is odd. But $u_2 = 2^3 - 1$, and therefore (3) is proved for all $e \geq 2$. It follows that $u_e$ is always odd.

Now, since $u = u_e = L_E$, we have $L_E \equiv 0 \pmod{u}$ and, similarly, $F_{2E} = F_E L_E \equiv 0 \pmod{u}$. Furthermore, (3) shows that $u + 1 = 2hE$ is a multiple of $2E$, and it follows that $F_{u+1} \equiv 0 \pmod{u}$.

Next, $\frac{1}{2}(u+1) = hE$ is an odd multiple of $E$, so that $L_{hE}$ is divisible by $L_E$ and $L_{hE} \equiv 0 \pmod{u}$. Thus, $L_{u+1} = L_{2hE} = (L_{hE})^2 - 2 \equiv -2 \pmod{u}$. From the identities $L_{u+2} + L_u = 5F_{u+1}$ and $L_{u+2} - L_u = L_{u+1}$, we have $2L_u = 5F_{u+1} - L_{u+1} \equiv 2 \pmod{u}$, giving $L_u \equiv 1 \pmod{u}$, since $u$ is odd.

**Remark:** Another proof of $L_u \equiv 1 \pmod{u}$, also based on formula (3) above, was given by A. Di Porto and P. Filipponi in their article "A Probabilistic Primality Test Based on the Properties of Certain Generalized Lucas Numbers" in *Lecture Notes in Computer Science* **330** (1988):211-223.

***Also solved by A. Dujella, H.-J. Seiffert, and the proposer.***

## FPP's and LPP's

**H-499** ***Proposed by Paul S. Bruckman, Edmonds, WA***
***(Vol. 33, no. 3, August 1995)***

Given $n$ a natural number, $n$ is a *Lucas pseudoprime* (LPP) if it is composite and satisfies the following congruence:

$$L_n \equiv 1 \pmod{n}. \tag{1}$$

If $\gcd(n, 10) = 1$, the *Jacobi symbol* $(5/n) = \varepsilon_n$ is given by the following:

$$\varepsilon_n = \begin{cases} 1 & \text{if } n \equiv \pm 1 \pmod{10}, \\ -1 & \text{if } n \equiv \pm 3 \pmod{10}. \end{cases}$$

Given $\gcd(n, 10) = 1$, $n$ is a *Fibonacci pseudoprime* (FPP) if it is composite and satisfies the following congruence:

$$F_{n-\varepsilon_n} \equiv 0 \pmod{n}. \tag{2}$$

Define the following sequences for $e = 1, 2, \ldots$:

$$u = u_e = F_{3^{e+1}} / F_{3^e}; \tag{3}$$

$$v = v_e = L_{3^{e+1}} / L_{3^e}; \tag{4}$$

$$w = w_e = F_{2\cdot 3^{e+1}} / F_{2\cdot 3^e} = uv. \tag{5}$$

Prove the following for all $e \geq 1$: (i) $u$ is a FPP and a LPP, provided it is composite; (ii) same statement for $v$; (iii) $w$ is a FPP but not a LPP.

***Solution by H.-J. Seiffert, Berlin, Germany***

We need the additional easily verifiable equations:

$$F_{3k} = 5F_k^3 + 3(-1)^k F_k, \tag{6}$$

$$L_{3k} = L_k^3 - 3(-1)^k L_k, \tag{7}$$

$$L_{2k} = L_k^2 - 2(-1)^k = 5F_k^2 + 2(-1)^k, \tag{8}$$

where $k$ is any integer, and the following propositions.

***Proposition 1:*** If $n$ is a composite positive integer such that $\gcd(n, 10) = 1$, then $n$ is a FPP and a LPP if and only if $F_{\frac{1}{2}(n-\varepsilon_n)} \equiv 0 \pmod{n}$ if $n \equiv 1 \pmod{4}$, and $L_{\frac{1}{2}(n-\varepsilon_n)} \equiv 0 \pmod{n}$ if $n \equiv 3 \pmod{4}$.

***Proof:*** This is just the result of H-496.

***Proposition 2:*** If $e$ is a positive integer, then $\frac{1}{2}L_{2\cdot 3^e}$ is an odd positive integer divisible by $3^{e+1}$.

***Proof:*** This is true for $e = 1$, since $\frac{1}{2}L_6 = 9$. Suppose that the statement holds for $e$, $e \in N$. Then we have $\frac{1}{2}L_{2\cdot 3^e} = 3^{e+1}m$, where $m$ is an odd positive integer. Equation (7) with $k = 2\cdot 3^e$ gives

$$\tfrac{1}{2}L_{2\cdot 3^{e+1}} = \tfrac{1}{2}\left(L_{2\cdot 3^e}^3 - 3L_{2\cdot 3^e}\right) = 3^{e+2}m(4\cdot 3^{2e+1}m^2 - 1),$$

showing that the statement holds for $e+1$. This completes the induction proof. Q.E.D.

***Proposition 3:*** If $k$ and $n$ are nonnegative integers, then we have $\gcd(L_k, L_{2kn}) \in \{1, 2\}$.

***Proof:*** From $L_{2kn} = L_k L_{k(2n-1)} - (-1)^k L_{2k(n-1)}$, it follows that

$$\gcd(L_k, L_{2kn}) \in \{\gcd(L_k, L_{2k}),\ \gcd(L_k, L_0)\}.$$

Since $L_{2k} = L_k^2 - 2(-1)^k$ and $L_0 = 2$, the desired result follows. Q.E.D.

Now we are able to prove the statements of the present proposal. From (6)-(8), we obtain

$$u_e + 1 = 5F_{3^e}^2 - 2 = L_{2\cdot 3^e} = L_{3^e}^2 + 2 = v_e - 1,\ e \in N. \tag{9}$$

Since $F_{3^e}$, $e \in N$, is even, it follows that $u = u_e \equiv -3 \pmod{10}$, $u \equiv 1 \pmod{4}$, $v = v_e \equiv -1 \pmod{10}$, and $v \equiv 3 \pmod{4}$, so that $\varepsilon_u = -1$ and $\varepsilon_v = 1$. Using Proposition 2, equations (9), and the well-known divisibility properties of the Fibonacci and Lucas numbers, we conclude that $F_{3^{e+1}} \mid F_{\frac{1}{2}(u+1)}$ and $L_{3^{e+1}} \mid L_{\frac{1}{2}(v-1)}$, which imply $F_{\frac{1}{2}(u+1)} \equiv 0 \pmod{u}$ and $L_{\frac{1}{2}(v-1)} \equiv 0 \pmod{v}$. Applying Proposition 1, we see that $u$ is a FPP and a LPP if it is composite, and that $v$ is a FPP and a LPP if it is composite. This solves (i) and (ii).

From what has been proved above, we have $w = uv \equiv 3 \pmod 4$, $w \equiv 3 \pmod{10}$, $\varepsilon_w = -1$, $w+1 = u(u+2)+1 = (u+1)^2$, and

$$F_{u+1} \equiv 0 \pmod{u} \quad \text{and} \quad F_{u+1} = F_{v-1} \equiv 0 \pmod{v}. \tag{10}$$

We note that (10) remains valid if $u$ or $v$ is a prime. Since $\gcd(u, v) = \gcd(u, u+2) = \gcd(u, 2) = 1$ and since $F_{u+1} \mid F_{(u+1)^2} = F_{w+1}$, from (10) we obtain $F_{w+1} \equiv 0 \pmod{w}$. Thus, $w$ is a FPP, since it is composite. However, $w$ is not a LPP. This can be seen as follows. Assume, by way of contradiction, that $w$ is a LPP. Then, since $w$ is a FPP as shown above, we would have $L_{\frac{1}{2}(w+1)} \equiv 0$ (mod $w$), by Proposition 1. It then would follows that

$$v \mid \gcd\left(L_{\frac{1}{2}(v-1)}, L_{\frac{1}{2}(w+1)}\right) = \gcd\left(L_{\frac{1}{2}(u+1)}, L_{\frac{1}{2}(u+1)^2}\right).$$

However, $\frac{1}{2}(u+1)^2$ is an even multiple of $\frac{1}{2}(u+1)$; thus, by Proposition 3, we have $v \in \{1, 2\}$. Clearly, this is a contradiction, since $v$ is obviously greater than 2. Hence, $w$ cannot be a LPP. This solves (iii).

***Also solved by L. A. G. Dresel and the proposer.***

**Belated Acknowledgment:** C. Georghiou solved H-486.

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