FIBONACCI PARTITIONS

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INTRODUCTION

Let $\{u_n\}$ be a strictly increasing sequence of natural numbers, so that $u_n \geq n$ for all n. Let

$$g(z) = \prod_{n\geq 1}(1 - z^{u_n}). \tag{1}$$

If $|z| < 1$, then

$$\left|\sum_{n\geq 1} z^{u_n}\right| \leq \sum_{n\geq 1}|z^{u_n}| = \sum_{n\geq}|z|^{u_n} \leq \sum_{n\geq 1}|z|^n = \frac{|z|}{1-|z|},$$

so the product in (1) converges absolutely to an analytic function without zeros on compact subsets of the unit disk. Let $g(z)$ have a Maclaurin series representation given by

$$g(z) = \sum_{n\geq 0} a_n z^n. \tag{2}$$

Let

$$f(z) = 1/g(z). \tag{3}$$

Then $f(z)$ is also an analytic function without zeros on compact subsets of the unit disk. We have

$$f(z) = \prod_{n\geq 1}(1 - z^{u_n})^{-1} = \sum_{n\geq 0} U_n z^n \quad (\text{with } U_0 = 1). \tag{4}$$

Definition 1: Let $r(n)$, $r_E(n)$, $r_0(n)$ denote, respectively, the number of partitions of n into distinct parts, evenly many distinct parts, oddly many distinct parts from $\{u_n\}$. Let $r(0) = r_E(0) = 1$, $r_0(0) = 0$. If $a_n = r_E(n) - r_0(n)$, then U_n is the number of partitions of n all of whose parts belong to $\{u_n\}$, that is, $f(z)$ is the generating function for $\{u_n\}$. Since $f(z) * g(z) = 1$, we obtain the recurrence relation:

$$\sum_{k=0}^{n} a_{n-k} U_k = 0 \quad (\text{for } n \geq 1). \tag{5}$$

This provides a way to determine the U_n, once the a_n are known. Now Definition 1 implies that

$$r_0(n) = r(n) - r_E(n); \tag{6}$$

hence,

$$a_n = 2r_E(n) - r(n). \tag{7}$$

Our original problem, namely, to determine U_n, has been reduced to determining the $r(n)$ and $r_E(n)$.

Several researchers have investigated the case where $\{u_n\}$ is the Fibonacci sequence. If we let $u_n = F_n$, as was done by Verner E. Hoggatt, Jr., & S. L. Basin [3], then an anomaly arises: since $F_1 = F_2 = 1$, it follows that 1 may occur twice as a summand in a partition of n into "distinct"

Fibonacci summands. We therefore prefer to let $u_n = F_{n+1}$, since the Fibonacci sequence is strictly increasing for $n \ge 2$. This is the approach taken by Klarner [4] and Carlitz [1]. Our algorithm for computing $r(n)$ is simpler and apparently more efficient than that of Carlitz.

Definition 2: The trivial partition of n consists of just n itself.

We shall use the following well-known properties of Fibonacci numbers:

$$F_m = F_{m-1} + F_{m-2}, \tag{8}$$

$$\sum_{k=1}^{m} F_k = F_{m+2} - 1, \tag{9}$$

$$\sum_{k=1}^{m} F_{2k} = F_{2m+1} - 1, \tag{10}$$

$$\sum_{k=2}^{m} F_{2k-1} = F_{2m} - 1, \tag{11}$$

Zeckendorf's Theorem (see [5]). (12)

Every natural number n has a unique representation:

$$n = \sum_{k=2}^{r} c_k F_k,$$

where $c_r = 1$, each $c_k = 0$ or 1, and $c_{k-1}c_k = 0$ for all k such that $3 \le k \le r$. Following Ferns [2], we call this the *minimal Fibonacci representation* of n.

More generally, if we drop the requirement that $c_{k-1}c_k = 0$, we obtain what will be called a Fibonacci representation of n. The c_k are called the digits of the representation. Now $r(n)$ denotes the number of distinct Fibonacci representations of n.

THE MAIN THEOREMS

Theorem 1: $r(F_m) = [\frac{1}{2}m]$ if $m \ge 2$.

Proof: (Induction on m) Since $r(F_2) = r(1) = 1 = [\frac{1}{2}(2)]$ and $r(F_3) = r(2) = 1 = [\frac{1}{2}(3)]$, Theorem 1 holds for $m = 2, 3$. Now suppose $m \ge 4$. Every nontrivial partition of F_m into distinct Fibonacci parts must include F_{m-1} as a part, since (9) implies that $\sum_{k=2}^{m-2} F_k = F_m - 2 < F_m$. Therefore, by (8), every nontrivial partition of F_m into distinct Fibonacci parts consists of F_{m-1}, plus the summands in such a partition of F_{m-2}. Therefore, $r(F_m) = 1 + r(F_{m-2})$ if $m \ge 4$. (The "1" in this formula corresponds to the trivial partition of F_m.) By the induction hypothesis, $r(F_{m-2}) = [\frac{1}{2}(m-2)]$. Thus, $r(F_m) = 1 + [\frac{1}{2}(m-2)] = [\frac{1}{2}m]$.

Remark: Essentially the same proof of Theorem 1 appears in [1] and [3].

Theorem 2: $r_E(F_m) = [\frac{1}{4}m]$ if $m \ge 2$.

Proof: (Induction on m) Since $r_E(F_2) = r_E(1) = 0 = [\frac{1}{4}(2)]$ and $r_E(F_3) = r_E(2) = 0 = [\frac{1}{4}(3)]$, Theorem 2 holds for $m = 2, 3$. Now suppose $m \ge 4$. As in the proof of Theorem 1, any partition of F_m into evenly many distinct Fibonacci parts must include F_{m-1} as a part, plus the summands in a partition of F_{m-2} into oddly many distinct Fibonacci parts. That is, $r_E(F_m) = r_0(F_{m-2})$. But (6), Theorem 1, and the induction hypothesis imply that $r_0(F_{m-2}) = r(F_{m-2}) - r_E(F_{m-2}) = [\frac{1}{2}(m-2)] - [\frac{1}{4}(m-2)] = [\frac{1}{4}m]$.

Theorem 3: Let $a(n) = a_n$. Then

$$a(F_m) = \begin{cases} 0 & \text{if } m \equiv 0, 1 \pmod 4, \\ -1 & \text{if } m \equiv 2, 3 \pmod 4. \end{cases}$$

Proof: From (7) and from Theorems 1 and 2, we have $a(F_m) = 2[\frac{1}{4}m] - [\frac{1}{2}m]$, from which the conclusion follows.

Having settled the case where n is a Fibonacci number, let us now consider the case where n is not a Fibonacci number. In the minimal Fibonacci representation, let $n = F_{k_1} + F_{k_2} + \cdots F_{k_r}$, where $r \geq 2$, $k_r \geq 2$, and $k_i - k_{i+1} \geq 2$ for all i with $1 \leq i \leq r-1$. Let $n_0 = n$, $n_i = n_{i-1} - F_{k_i}$ for $1 \leq i \leq r$. In particular, $n_1 = n - F_{k_1}$, $n_{r-1} = F_{k_r}$, $n_r = 0$. Given any Fibonacci representation of n, define the *initial segment* as the first $k_1 - k_2$ digits, while the *terminal segment* consists of the remaining digits. In the minimal Fibonacci representation of n, the initial segment consists of a 1 followed by $k_1 - k_2 - 1$ 0's, while the terminal segment starts with 10. Fibonacci representations of n may be obtained as follows:

Type I: Arbitrary combinations of Fibonacci representations of the integers corresponding to the initial and terminal segments in the minimal Fibonacci representation of n;

Type II: Suppose that in a nonminimal Fibonacci representation of n the initial segment ends in 10 while the terminal segment starts with 0. If this 100 block, which is partly in the initial segment and partly in the terminal segment, is replaced by 011, a new Fibonacci representation of n is obtained.

Lemma 1: Every Fibonacci representation of n that includes F_{k_1} as a part has an initial segment which agrees with that of the minimal Fibonacci representation.

Proof: If n has a Fibonacci representation that includes F_{k_1} as a part but differs from the minimal Fibonacci representation, then $n = F_{k_1} + F_j + \cdots$, where $j > k_2$. But $n \leq F_{k_1} + F_{k_2} + F_{k_2-2} + F_{k_2-4} + \cdots \leq F_{k_1} + F_{k_2+1} - 1$ by (10) and (11). Now $F_{k_1} + F_j \leq n \leq F_{k_1} + F_{k_2+1}$, which implies $F_j < F_{k_2+1}$; hence, $j \leq k_2$, an impossibility.

Lemma 2: Let $\bar{r}(n)$ be the number of Fibonacci representations of n that do not include F_{k_1} as a part. Then $\bar{r}(n) = r(n) - r(n_1)$.

Proof: If n is a Fibonacci number, then the conclusion follows from Definitions 1 and 2. Otherwise, by hypothesis, $r(n) - \bar{r}(n)$ is the number of Fibonacci representations of n that do include F_{k_1} as a part. By Lemma 1, the initial segment of such a representation is unique, and consists of a 1 followed by $k_1 - k_2 - 1$ 0's. Since the terminal segment is unrestricted, the number of such Type I representations is $1 * r(n_1) = r(n_1)$. Type II representations are excluded here, since they can only arise when the initial segment has a nonminimal representation. Therefore, we have: $r(n) - \bar{r}(n) = r(n_1)$, from which the conclusion follows.

Lemma 3: Let $\bar{r}_E(n)$ denote the number of partitions of n into evenly many distinct Fibonacci numbers, not including F_{k_1} as a part. Then $\bar{r}_E(n) = r_E(n) - r_0(n_1)$.

Proof: The proof of Lemma 3 is similar to that of Lemma 2, and is therefore omitted.

Theorem 4: $r(n) = \begin{cases} \tfrac{1}{2}(k_1 - k_2 + 1)r(n_1) & \text{if } k_1 - k_2 \text{ is odd,} \\ (1 + \tfrac{1}{2}(k_1 - k_2))r(n_1) - r(n_2) & \text{if } k_1 - k_2 \text{ is even.} \end{cases}$

Proof: Let $m = k_1$, $h = k_2$. Recall that the initial segment of the minimal Fibonacci representation of n consists of a 1 followed by $m-h-1$ 0's. Viewed by itself, this initial segment corresponds to the minimal Fibonacci representation of F_{m-h+1}. By Theorem 1, the number of Fibonacci representations of the initial segment is $r(F_{m-h+1}) = [\tfrac{1}{2}(m-h+1)]$. The number of Fibonacci representations of the terminal segment is by definition $r(n_1)$. Therefore, the number of Type I Fibonacci representations of n is $[\tfrac{1}{2}(m-h+1)]r(n_1)$.

If $m-h$ is odd, then the initial segment in the minimal Fibonacci representation of n consists of a 1 followed by evenly many 0's. Therefore, each Fibonacci representation of F_{m-h+1} (the integer corresponding to the initial segment) ends in 00 or 11. Thus, Type II Fibonacci representations of n cannot arise, so that $r(n) = [\tfrac{1}{2}(m-h+1)]r(n_1) = \tfrac{1}{2}(m-h+1)r(n_1)$.

If $m-h$ is even, then the initial segment in the minimal Fibonacci representation of n consists of a 1 followed by oddly many 0's. Therefore, F_{m-h+1} has a unique Fibonacci representation ending in 10. By Lemma 2, the integer corresponding to the terminal segment, namely n_1, has $\bar{r}(n_1) = r(n_1) - r(n_2)$ Fibonacci representations that start with digit 0. Thus, we have $r(n_1) - r(n_2)$ Type II Fibonacci representations of n. Therefore, $r(n_1) = [\tfrac{1}{2}(m-h+1)]r(n_1) + r(n_1) - r(n_2)$. Simlifying, we get $r(n) = (1 + \tfrac{1}{2}(m-h))r(n_1) - r(n_2)$.

Theorem 5:

(a) If $k_1 - k_2 \equiv 3 \pmod 4$, then $r_E(n) = \tfrac{1}{4}(k_1 - k_2 + 1)r(n_1)$.

(b) If $k_1 - k_2 \equiv 1 \pmod 4$, then $r_E(n) = \tfrac{1}{4}(k_1 - k_2 + 3)r(n_1) - r_E(n_1)$.

(c) If $k_1 - k_2 \equiv 2 \pmod 4$, then $r_E(n) = \tfrac{1}{4}(k_1 - k_2 + 2)r(n_1) + r_E(n_2) - r(n_2)$.

(d) If $k_1 - k_2 \equiv 0 \pmod 4$, then $r_E(n) = (1 + \tfrac{1}{4}(k_1 - k_2))r(n_1) - r_E(n_1) - r_E(n_2)$.

Proof: Let $m = k_1$, $h = k_2$. Let $b(n)$ and $c(n)$ denote, respectively, the numbers of Type I and Type II representations of n as a sum of evenly many distinct Fibonacci numbers, so that $r_E(n) = b(n) + c(n)$. A Fibonacci representation of n has evenly many parts if and only if the number of 1's in the initial segment has the same parity as the number of 1's in the terminal segment. Thus,

$$\begin{aligned} b(n) &= r_E(F_{m-h+1})r_E(n_1) + r_0(F_{m-h+1})r_0(n_1) \\ &= [\tfrac{1}{4}(m-h+1)]r_E(n_1) + ([\tfrac{1}{2}(m-h+1)] - [\tfrac{1}{4}(m-h+1)])(r(n_1) - r_E(n_1)) \\ &= ([\tfrac{1}{2}(m-h+1)] - [\tfrac{1}{4}(m-h+1)])r(n_1) + (2[\tfrac{1}{4}(m-h+1)] - [\tfrac{1}{2}(m-h+1)])r_E(n_1). \end{aligned}$$

If $m-h \equiv 0$ or $3 \pmod 4$, then $[\tfrac{1}{2}(m-h+1)] = 2[\tfrac{1}{4}(m-h+1)]$, so $b(n) = [\tfrac{1}{4}(m-h+1)]r(n_1)$.

If $m-h \equiv 1$ or $2 \pmod 4$, then $[\tfrac{1}{2}(m-h+1)] = 1 + 2[\tfrac{1}{4}(m-h+1)]$, so $b(n) = (1 + [\tfrac{1}{4}(m-h+1)])r(n_1) - r_E(n_1)$.

If $m-h$ is odd, then, as in the proof of Theorem 4, no Type II Fibonacci representations of n can occur, that is, $c(n)=0$. Upon simplifying, we obtain:

(a) If $m-h \equiv 3 \pmod 4$, then $r_E(n) = \frac{1}{4}(m-h+1)r(n_1)$;

(b) If $m-h \equiv 1 \pmod 4$, then $r_E(n) = \frac{1}{4}(m-h+3)r(n_1) - r_E(n_1)$.

If $m-h$;is even, then, as in the proof of Theorem 4, the integer corresponding to the initial segment has a unique Fibonacci representation ending in 10, so that Type II Fibonacci representations of n do occur. A Type II Fibonacci representation will have evenly many 1's if and only if the number of 1's in the initial and terminal segments differ in parity.

If $m-h \equiv 2 \pmod 4$, then the unique Fibonacci representation of the integer corresponding to the initial segment that ends in 10 has an odd number of 1's. Therefore,

$$c(n) = \bar{r}_E(n_1) = r_E(n_1) - r_0(n_2).$$

Thus,

$$\begin{aligned} r_E(n) &= b(n) + c(n) \\ &= (1 + [\tfrac{1}{4}(m-h+1)])r(n_1) - r_E(n_1) + r_E(n_1) - r_0(n_2) \\ &= \tfrac{1}{4}(m-h+2)r(n_1) + r_E(n_2) - r(n_2). \end{aligned}$$

This proves ***(c)***.

If $m-h \equiv 0 \pmod 4$, then the unique Fibonacci representation of the integer corresponding to the initial segment that ends in 10 has an even number of 1's. Therefore,

$$\begin{aligned} c(n) = \bar{r}_0(n_1) &= \bar{r}(n_1) - \bar{r}_E(n_1) \\ &= r(n_1) - r(n_2) - (r_E(n_1) - r_0(n_2)) \\ &= r(n_1) - r_E(n_1) - r_E(n_2). \end{aligned}$$

But $b(n) = [\frac{1}{4}(m-h+1)]r(n_1) = \frac{1}{4}(m-h)r(n_1)$, so

$$r_E(n) = b(n) + c(n) = (1 + \tfrac{1}{4}(m-h))r(n_1) - r_E(n_2).$$

This proves ***(d)***.

Theorem 6: If n is not a Fibonacci number, then

$$a(n) = \begin{cases} -a(n_1) - a(n_2) & \text{if } k_1 - k_2 \equiv 0 \pmod 4, \\ -a(n_1) & \text{if } k_1 - k_2 \equiv 1 \pmod 4, \\ a(n_2) & \text{if } k_1 - k_2 \equiv 2 \pmod 4, \\ 0 & \text{if } k_1 - k_2 \equiv 3 \pmod 4. \end{cases}$$

Proof: This follows from (7) and from Theorems 4 and 5.

Theorem 7: $a(n) = \begin{cases} 0 & \text{if } r(n) \text{ is even}, \\ \pm 1 & \text{if } r(n) \text{ is odd}. \end{cases}$

Proof: If n is a Fibonacci number, then the conclusion follows from Theorems 1 and 3. If n is not a Fibonacci number, then we will use induction. Note that (7) implies $a(n) \equiv r(n) \pmod 2$. Therefore, it suffices to show that $|a(n)| \le 1$. By Theorem 6 and the induction hypothesis, this is true, except possibly when $k_1 - k_2 \equiv 0 \pmod 4$. In this case, we have $a(n) = -a(n_1) - a(n_2)$. Again by Theorem 6 we have:

$$a(n_1) = \begin{cases} -a(n_2) - a(n_3) & \text{if } k_2 - k_3 \equiv 0 \pmod 4, \\ -a(n_2) & \text{if } k_2 - k_3 \equiv 1 \pmod 4, \\ a(n_3) & \text{if } k_2 - k_3 \equiv 2 \pmod 4, \\ 0 & \text{if } k_2 - k_3 \equiv 3 \pmod 4. \end{cases}$$

Therefore, we have

$$a(n) = \begin{cases} a(n_3) & \text{if } k_2 - k_3 \equiv 0 \pmod 4, \\ 0 & \text{if } k_2 - k_3 \equiv 1 \pmod 4, \\ -a(n_2) - a(n_3) & \text{if } k_2 - k_3 \equiv 2 \pmod 4, \\ -a(n_2) & \text{if } k_2 - k_3 \equiv 3 \pmod 4. \end{cases}$$

Thus, $|a(n)| \le 1$ except, possibly, when $k_2 - k_3 \equiv 2 \pmod 4$. In the latter case, we evaluate $a(n_2)$ using Theorem 6. We then see that $|a(n)| \le 1$ except, possibly, when $k_3 - k_4 \equiv 2 \pmod 4$, in which case $a(n) = -a(n_3) - a(n_4)$. If $|a(n)| > 1$, then we would have an infinite sequence: $n > n_1 > n_2 > n_3 > \cdots$. This is impossible, so we must have $|a(n)| \le 1$ for all n.

Theorem 8: $r(n) = 1$ if and only if $n = F_m - 1$ for some $m \ge 2$; if so, then

$$a(n) = \begin{cases} 1 & \text{if } m \equiv 1, 2 \pmod 4, \\ -1 & \text{if } m \equiv 0, 3 \pmod 4. \end{cases}$$

Proof: First, suppose that $n = F_m - 1$. By (10) and (11), we have

$$n = \sum_{k=1}^{[\frac{1}{2}m-1]} F_{m+1-2k}.$$

This is the minimal Fibonacci representation of n (since the condition $c_{j-1}c_j = 0$ holds) and consists of alternating 1's and 0's. Since no two consecutive 0's appear, this Fibonacci representation is also maximal; hence, is unique, that is, $r(n) = 1$. Conversely, if $r(n) = 1$, then the unique Fibonacci representation of n cannot contain consecutive 0's, and thus must consist of alternating 1's and 0's. Therefore, for some m, we have $n = F_{m-1} + F_{m-3} + F_{m-5} + \cdots$. Now (10) and (11) imply $n = F_m - 1$. If $m = 4j+1$ or $4j+2$ for some j, then the unique Fibonacci representation of n has $2j$ summands. Thus, $a(n) = 1$ if $m \equiv 1, 2 \pmod 4$. On the other hand, if $m = 4j$ or $4j-1$, then the unique Fibonacci representation of n has $2j-1$ summands. Therefore, $a(n) = -1$ if $m = 0, 3 \pmod 4$.

Theorem 9: There are arbitrarily long sequences of integers n such that $a(n) = 0$.

Proof: If $F_m + F_{m-3} \le n \le F_m + F_{m-2} - 1$, then the minimal Fibonacci representation of n is $n = F_m + F_{m-3} + \cdots$. Therefore, Theorem 6 implies that $a(n) = 0$. The number of integers satisfying the above inequality is $F_{m-2} - F_{m-3} = F_{m-4}$. For any given h, we can find $m \ge 6$ such that $F_{m-4} \ge h$. Thus, we are done.

Remark: With a little additional effort, one can also show that $a(F_m + F_{m-3} - 1) = 0$.

Using (5) as well as Theorems 3, 4, and 6, one can compute $r(n), a(n)$, and $U(n)$ for any n. Table 1 lists the results of these computations for $1 \le n \le 100$.

TABLE 1

n	$r(n)$	$a(n)$	$U(n)$	n	$r(n)$	$a(n)$	$U(n)$
1	1	–1	1	51	3	1	4017
2	1	–1	2	52	4	0	4367
3	2	0	3	53	4	0	4737
4	1	1	4	54	1	1	5134
5	2	0	6	55	5	–1	5564
6	2	0	8	56	4	0	6016
7	1	1	10	57	4	0	6504
8	3	–1	14	58	7	1	7025
9	2	0	17	59	3	–1	7575
10	2	0	22	60	6	0	8171
11	3	1	27	61	6	0	8791
12	1	–1	33	62	3	–1	9466
13	3	–1	41	63	8	0	10183
14	3	1	49	64	5	1	10936
15	2	0	59	65	5	1	11744
16	4	0	71	66	7	–1	12599
17	2	0	83	67	2	0	13502
18	3	1	99	68	6	0	14471
19	3	–1	115	69	6	0	15486
20	1	–1	134	70	4	0	16568
21	4	0	157	71	8	0	17715
22	3	1	180	72	4	0	18921
23	3	1	208	73	6	0	20207
24	5	–1	239	74	6	0	21559
25	2	0	272	75	2	0	22987
26	4	0	312	76	7	1	24506
27	4	0	353	77	5	–1	26094
28	2	0	400	78	5	–1	27782
29	5	1	453	79	8	0	29558
30	3	–1	509	80	3	1	31425
31	3	–1	573	81	6	0	33405
32	4	0	642	82	6	0	35478
33	1	1	717	83	3	1	37664
34	4	0	803	84	7	–1	39973
35	4	0	892	85	4	0	42386
36	3	1	993	86	4	0	44939
37	6	0	1102	87	5	1	47613
38	3	–1	1219	88	1	–1	50421
39	5	–1	1350	89	5	–1	53384
40	5	1	1489	90	5	1	56478
41	2	0	1640	91	4	0	59735
42	6	0	1808	92	8	0	63154
43	4	0	1983	93	4	0	66727
44	4	0	2178	94	7	1	70492
45	6	0	2386	95	7	–1	74422
46	2	0	2609	96	3	–1	78543
47	5	1	2854	97	9	1	82871
48	5	–1	3113	98	6	0	87383
49	3	–1	3393	99	6	0	92122
50	6	0	3697	100	9	–1	97075

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REFERENCES

1. L. Carlitz. "Fibonacci Representations." *The Fibonacci Quarterly* **6.4** (1968):193-220.
2. H. H. Ferns. "On the Representation of Integers as Sums of Distinct Fibonacci Numbers." *The Fibonacci Quarterly* **3.1** (1965):21-30.
3. V. E. Hoggatt, Jr., & S. L. Basin. "Representations by Complete Sequences—Part I." *The Fibonacci Quarterly* **1.3** (1963):1-14.
4. D. Klarner. "Partitions of N into Distinct Fibonacci Numbers." *The Fibonacci Quarterly* **6.4** (1968):235-43.
5. C. G. Lekkerkerker. "Voorstelling van natuurlyke getallen door een som van Fibonacci." *Simon Stevin* **29** (1951-1952):190-95.

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