SUMMATION OF RECIPROCALS IN CERTAIN SECOND-ORDER RECURRING SEQUENCES

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1. INTRODUCTION

We consider the sequence $\{W_n\} = \{W_n(a,b;\,P,Q)\}$ of integers defined by

$$W_0 = a,\; W_1 = b,\; W_n = PW_{n-1} - QW_{n-2} \quad (n \geq 2), \tag{1.1}$$

where a, b, P, and Q are integers, with $PQ \neq 0$. Particular cases of $\{W_n\}$ are the sequences $\{U_n\}$ of Fibonacci and $\{V_n\}$ of Lucas defined by $U_n = W_n(0,1;\,P,Q)$ and $V_n = W_n(2,P;\,P,Q)$. In the sequel we shall suppose that $\Delta = P^2 - 4Q > 0$. It is readily proven [6] that

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}, \tag{1.2}$$

where $\alpha = (P + \sqrt{\Delta})/2$, $\beta = (P - \sqrt{\Delta})/2$, $A = b - \beta a$, and $B = b - \alpha a$. Following Horadam [6], we define the number e_w by $e_w = AB = b^2 - Pab + Qa^2$. It is clear that $e_u = 1$ and $e_v = -\Delta = -(\alpha - \beta)^2$, where e_u and e_v are associated with the Fibonacci and Lucas sequences. By means of the Binet form (1.2), one can easily prove the Catalan relation

$$W_n^2 - W_{n-1}W_{n+1} = e_w Q^{n-1}. \tag{1.3}$$

Notice that

$$\alpha > 1 \text{ and } \alpha > |\beta|, \text{ if } P > 0, \tag{1.4}$$

and that

$$\beta < -1 \text{ and } |\beta| > |\alpha|, \text{ if } P < 0. \tag{1.5}$$

By (1.4) and (1.5), it is clear that $U_n \neq 0$ for $n \geq 1$ and that $V_n \neq 0$ for $n \geq 0$. More generally, there exists an integer p such that $W_p = 0$ if and only if $W_n = W_{p+1}U_{n-p}$ for every integer n. By (1.4) and (1.5), we obtain

$$W_n \simeq \frac{A}{\alpha - \beta}\alpha^n, \text{ if } P > 0 \text{ and } W_n \simeq \frac{-B}{\alpha - \beta}\beta^n, \text{ if } P < 0. \tag{1.6}$$

The purpose of this paper is to investigate the infinite sums

$$S_k = \sum_{n=1}^{+\infty} \frac{Q^n}{W_n W_{n+k}} \quad \text{and} \quad T_k = \sum_{n=1}^{+\infty} \frac{1}{W_n W_{n+k}},$$

where k is a positive integer. We shall suppose that $W_n \neq 0$ for $n \geq 1$ (see the remark above) and that $e_w = AB \neq 0$ (which means that $\{W_n\}$ is not a purely geometric sequence). By (1.4) and (1.5), use of the ratio test shows that the series S_k and T_k are absolutely convergent. Notice that $S_k = T_k$, when $Q = 1$.

More generally, let $\pi(n) = m + sn$ be an arithmetical progression, with $m \geq 0$ and $s \geq 1$. We shall examine the sums

$$S_{k,\pi} = \sum_{n=1}^{+\infty} \frac{Q^{\pi(n)}}{W_{\pi(n)}W_{\pi(n+k)}} \quad \text{and} \quad T_{k,\pi} = \sum_{n=1}^{+\infty} \frac{1}{W_{\pi(n)}W_{\pi(n+k)}}.$$

By the way, we shall also obtain a symmetry property (Theorem 1) that generalizes a recent result of Good [5].

Remark 1: Notice that $S_{k,\pi} = T_{k,\pi}$ when $Q = 1$ and that $S_{k,\pi} = (-1)^m T_{k,\pi}$ when $Q = -1$ and s is even.

2. MAIN RESULTS

Theorem 1: We have

$$U_k \sum_{n=1}^{m} \frac{Q^n}{W_n W_{n+k}} = U_m \sum_{n=1}^{k} \frac{Q^n}{W_n W_{n+m}},$$

where k and m are nonnegative integers.

Theorem 2: If $P > 0$, then

$$S_k = \frac{1}{e_w U_k}\left[\sum_{r=1}^{k} \frac{W_{r+1}}{W_r} - k\alpha\right]. \tag{2.1}$$

If $P < 0$, replace α by β in the right member.

Theorem 2': If $P > 0$ or if $P < 0$ and s is even, then

$$S_{k,\pi} = \frac{1}{e_w U_s U_{sk}}\left[\sum_{r=1}^{k} \frac{W_{\pi(r+1)}}{W_{\pi(r)}} - k\alpha^s\right]. \tag{2.2}$$

If $P < 0$ and s is odd, replace α^s by β^s in the right member.

Theorem 3: If $P > 0$, then

$$AU_k T_k = (1 - Q^k)\sum_{r=1}^{+\infty} \frac{1}{\alpha^r W_r} + Q^k \sum_{r=1}^{k} \frac{1}{\alpha^r W_r}. \tag{2.3}$$

If $P < 0$, replace A by B in the left member and α by β in the right member.

Corollary 1: If $Q = -1$, then

$$T_{2k} = \frac{1}{U_{2k}} \sum_{r=1}^{k} \frac{1}{W_{2r} W_{2r-1}} \tag{2.4}$$

and

$$T_{2k+1} = \frac{1}{U_{2k+1}}\left[T_1 - \sum_{r=1}^{k} \frac{1}{W_{2r} W_{2r+1}}\right]. \tag{2.5}$$

Corollary 2: If $Q = -1$ and s is odd, then

$$T_{2k,\pi} = \frac{U_s}{U_{2ks}} \sum_{r=1}^{k} \frac{1}{W_{\pi(2r)} W_{\pi(2r-1)}} \tag{2.6}$$

and

$$T_{2k+1,\pi} = \frac{U_s}{U_{(2k+1)s}}\left[T_{1,\pi} - \sum_{r=1}^{k}\frac{1}{W_{\pi(2r)}W_{\pi(2r+1)}}\right]. \tag{2.7}$$

Remark 2: If $Q=-1$, $k=1$, and $W_n = U_n$ or V_n, then Theorem 3 is Lemma 2 in [1].

Remark 3: Theorem 1 shows that S_k is a rational number if and only if α is rational or, equivalently, if and only if Δ is a perfect square. Corollary 1 shows that, in the case $Q=-1$, T_{2k} is rational, while T_{2k+1} is rational if and only if T_1 is rational. Notice that, even in the usual case $W_n = W_n(0,1;1,-1) = F_n$, the value and the arithmetical nature of T_1 is unknown. One can obtain similar results for the numbers $S_{k,\pi}$ and $T_{k,\pi}$.

Theorem 1 is given by Good [5] in the case $Q=-1$. Theorem 2′ was first obtained by Lucas [8, p. 198] in the case $k=1$, $W_n = U_n$ or V_n. The same results were rediscovered by Popov [11]. Brousseau [3] proved Theorem 2 for $W_n = F_n$ and he gave numerical examples of Corollary 1. Good [5] proved Theorem 2 in the case $Q=-1$. In [2], [7], and [9], one can find variants of Theorem 2′ applied to Fibonacci, Lucas, Pell, and Chebyshev polynomials.

3. PRELIMINARIES

In the sequel, we shall need the following lemmas.

Lemma 1: For integers $n \ge 0$ and $k \ge 0$

$$\begin{cases} W_{n+k} - \beta^k W_n = A\alpha^n U_k, & (3.1)\\ W_{n+k} - \alpha^k W_n = B\beta^n U_k. & (3.2)\end{cases}$$

Proof: Using Binet form (1.2), the result is immediate.

Lemma 2: For integers $k \ge 1$,

$$\sum_{r=1}^{k}\frac{\beta^r}{W_r} = \frac{1}{B}\left[\sum_{r=1}^{k}\frac{W_{r+1}}{W_r} - k\alpha\right], \tag{3.3}$$

$$\sum_{r=1}^{k}\frac{\alpha^r}{W_r} = \frac{1}{A}\left[\sum_{r=1}^{k}\frac{W_{r+1}}{W_r} - k\beta\right]. \tag{3.4}$$

Proof: We prove only (3.3); the proof of (3.4) is similar. By (3.2), where $n=r$ and $k=1$, we have

$$\sum_{r=1}^{k}\frac{\beta^r}{W_r} = \frac{1}{B}\sum_{r=1}^{k}\frac{W_{r+1}-\alpha W_r}{W_r} = \frac{1}{B}\left[\sum_{r=1}^{k}\frac{W_{r+1}}{W_r} - k\alpha\right].$$

Lemma 3: If $Q=-1$, we have, for $k \ge 1$,

$$\sum_{r=1}^{k}\frac{1}{\alpha^r W_r} = A\sum_{r=1}^{k}\frac{1}{W_{2r}W_{2r-1}}, \tag{3.5}$$

$$\sum_{r=2}^{2k+1} \frac{1}{\alpha^r W_r} = A\sum_{r=1}^{k} \frac{1}{W_{2r}W_{2r+1}}. \tag{3.6}$$

One can obtain two similar formulas by replacing α by β and A by B.

Proof: We prove only (3.5). Since $Q=-1$, we have $\alpha^r\beta^r=(-1)^r$ for $k\ge 1$; thus,

$$\begin{aligned}\sum_{r=1}^{2k}\frac{1}{\alpha^r W_r} &= \frac{1}{B}\sum_{r=1}^{2k}\frac{(-1)^r\beta^r B}{W_r} = \frac{1}{B}\sum_{r=1}^{2k}(-1)^r\frac{W_{r+1}-\alpha W_r}{W_r}, \quad \text{by (3.2)}\\ &= \frac{1}{B}\sum_{r=1}^{2k}(-1)^r\frac{W_{r+1}}{W_r} = \frac{1}{B}\sum_{r=1}^{k}\left(\frac{-W_{2r}}{W_{2r-1}}+\frac{W_{2r+1}}{W_{2r}}\right)\\ &= \frac{1}{B}\sum_{r=1}^{k}\frac{W_{2r+1}W_{2r-1}-W_{2r}^2}{W_{2r}W_{2r-1}} = \frac{1}{B}\sum_{r=1}^{k}\frac{-e_w(-1)^{2r-1}}{W_{2r}W_{2r-1}}, \quad \text{by (1.3)}\\ &= A\sum_{r=1}^{k}\frac{1}{W_{2r}W_{2r-1}}, \quad \text{since } e_w = AB.\end{aligned}$$

Lemma 4: Let $\{a_n\}$ be a sequence of numbers and $\{b_{n,k}\}$ be the sequence defined by

$$b_{n,k} = a_n - a_{n+k}, \quad k\ge 0. \tag{3.7}$$

For every $m\ge 0$ and $k\ge 0$, we then have

$$\sum_{n=1}^{m} b_{n,k} = \sum_{n=1}^{k} b_{n,m}. \tag{3.8}$$

Proof: Without loss of generality, we assume $m>k$. By (3.7) we get

$$\begin{aligned}\sum_{n=1}^{m} b_{n,k} &= (a_1+\cdots+a_m)-(a_{k+1}+\cdots+a_{m+k})\\ &= (a_1+\cdots+a_k)+(a_{k+1}+\cdots+a_m)-(a_{k+1}+\cdots+a_m)-(a_{m+1}+\cdots+a_{m+k})\\ &= (a_1+\cdots+a_k)-(a_{m+1}+\cdots+a_{m+k}) = \sum_{n=1}^{k} b_{n,m}.\end{aligned}$$

4. PROOF OF THEOREMS 1, 2, AND 2′

We get by (3.1) that

$$\frac{\beta^n}{W_n} - \frac{\beta^{n+k}}{W_{n+k}} = \frac{AQ^nU_k}{W_nW_{n+k}}. \tag{4.1}$$

Putting $a_n=\beta^n/W_n$ and $b_{n,k}=AQ^nU_k/W_nW_{n+k}$, we see by (4.1) that $b_{n,k}=a_n-a_{n+k}$. Theorem 1 follows immediately by this and Lemma 4.

Assuming now that $P>0$ and letting $n=1,2,\ldots,N$, where $N\ge k$, we obtain

$$AU_k\sum_{n=1}^{N}\frac{Q^n}{W_nW_{n+k}} = \sum_{r=1}^{k}\frac{\beta^r}{W_r} - \sum_{r=N+1}^{N+k}\frac{\beta^r}{W_r}.$$

Now, by (1.6) we have

$$\frac{\beta^r}{W_r} \simeq \frac{\alpha-\beta}{A}\left(\frac{\beta}{\alpha}\right)^r,$$

and since $\alpha > |\beta|$, the last sum in the right member vanishes as $N \to +\infty$. Thus, by (3.3),

$$AU_k \sum_{n=1}^{+\infty} \frac{Q^n}{W_n W_{n+k}} = \sum_{r=1}^{k} \frac{\beta^r}{W_r} = \frac{1}{B}\left[\sum_{r=1}^{k} \frac{W_{r+1}}{W_r} - k\alpha\right],$$

and the conclusion follows from this, since $e_w = AB$. If $P < 0$, replace β by α in the left member of (4.1) and A by B in the right member. Using (3.2) and (3.4) and recalling that $|\beta| > |\alpha|$ in this case, the end of the proof is similar.

Let us examine some particular cases. If $W_n = U_n$ (respectively V_n) and since $e_u = 1$ (respectively $e_v = -\Delta$), we get that

$$\sum_{n=1}^{+\infty} \frac{Q^n}{U_n U_{n+k}} = \frac{1}{U_k}\left[\sum_{r=1}^{k} \frac{U_{r+1}}{U_r} - k\alpha\right] \tag{4.2}$$

and

$$\sum_{n=1}^{+\infty} \frac{Q^n}{V_n V_{n+k}} = \frac{1}{\Delta U_k}\left[k\alpha - \sum_{r=1}^{k} \frac{V_{r+1}}{V_r}\right] \tag{4.3}$$

when $P > 0$.

If $P < 0$, replace α by β in the above formulas.

We turn now to the proof of Theorem 2′. Let us consider a second-order recurring sequence $\{W_n'\}$ (see [4] and [10]) satisfying

$$W_n' = P'W_{n-1}' - Q'W_{n-2}',\ n \geq 2, \tag{4.4}$$

where $P' = \alpha^s + \beta^s = V_s$ and $Q' = \alpha^s\beta^s = Q^s$. Notice that $P' > 0$ if and only if $P > 0$ or if $P < 0$ and s is even. The Fibonacci sequence associated with the recurrence (4.4) is defined by

$$U_n' = \frac{\alpha^{sn} - \beta^{sn}}{\alpha^s - \beta^s} = \frac{U_{sn}}{U_s}. \tag{4.5}$$

On the other hand, we have

$$W_{\pi(n)} = W_{m+sn} = \frac{A'\alpha^{sn} - B'\beta^{sn}}{\alpha - \beta},$$

where $A' = A\alpha^m$ and $B' = B\beta^m$. If $\{W_n'\}$ is the solution of (4.4) defined by $W_n' = \frac{A'\alpha^{sn} - B'\beta^{sn}}{\alpha^s - \beta^s}$, we have

$$W_n' = \frac{W_{\pi(n)}}{U_s}. \tag{4.6}$$

It follows by Theorem 2 applied to $\{W_n'\}$ that, if $P' > 0$,

$$\sum_{n=1}^{+\infty} \frac{Q^{sn}}{W_n' W_{n+k}'} = \frac{1}{e_{w'} U_k'}\left[\sum_{r=1}^{k} \frac{w_{r+1}'}{w_r'} - k\alpha^s\right]. \tag{4.7}$$

Using (4.5) and (4.6) and noticing that $e_{w'} = A'B' = AB\alpha^m\beta^m = e_w Q^m$, we easily deduce (2.2) from (4.7). If $P' < 0$, replace α^s by β^s in the right member of (4.7).

5. PROOF OF THEOREM 3 AND COROLLARIES 1 AND 2

Supposing first that $P>0$, we get by (3.1) that

$$\frac{1}{\alpha^n W_n} - \frac{Q^k}{\alpha^{n+k} W_{n+k}} = \frac{AU_k}{W_n W_{n+k}}. \tag{5.1}$$

Letting $n = 1, 2, \ldots, N$, where $N \geq k$, and summing, we obtain

$$\begin{aligned} AU_k \sum_{n=1}^{N} \frac{1}{W_n W_{n+k}} &= \sum_{r=1}^{k} \frac{1}{\alpha^r W_r} + (1-Q^k) \sum_{r=k+1}^{N} \frac{1}{\alpha^r W_r} - Q^k \sum_{r=N+1}^{N+k} \frac{1}{\alpha^r W_r} \\ &= (1-Q^k) \sum_{r=1}^{N} \frac{1}{\alpha^r W_r} + Q^k \sum_{r=1}^{k} \frac{1}{\alpha^r W_r} - Q^k \sum_{r=N+1}^{N+k} \frac{1}{\alpha^r W_r}. \end{aligned}$$

The first sum in the right member converges as $N \to +\infty$ since $\alpha^r W_r \simeq \frac{A}{\alpha-\beta}\alpha^{2r}$, where $\alpha > 1$. We also see that the last sum vanishes when $N \to +\infty$. This concludes the proof of Theorem 3 when $P > 0$. If $P < 0$, the proof is similar.

Notice that the first term in the right member of (2.3) vanishes if and only if $Q = 1$ (in which case $S_k = T_k$) or $Q = -1$ and k is even. The series $\sum_{r=1}^{+\infty} \frac{1}{\alpha^r W_r}$ seems difficult to evaluate. If $Q = -1$ and if $W_n = U_n$ or $W_n = V_n$, this series can be expressed with the help of the Lambert series [1, Lemma 3]. If $Q = 1$, it does not appear in (2.3). This fact explains why Melham and Shannon [9, p. 199] obtain formulas that do not involve Lambert series.

If $Q = -1$ and k is even, then (2.3) becomes

$$AU_{2k}T_{2k} = \sum_{r=1}^{2k} \frac{1}{\alpha^r W_r} = A \sum_{r=1}^{k} \frac{1}{W_{2r} W_{2r-1}}$$

by (3.5), when $P > 0$. This concludes the proof of (2.4). If $P < 0$, the proof is similar.

On the other hand, put $Q = -1$ and replace k by $2k+1$ in (2.2) to obtain

$$AU_{2k+1}T_{2k+1} = 2\sum_{r=1}^{+\infty} \frac{1}{\alpha^r W_r} - \sum_{r=1}^{2k+1} \frac{1}{\alpha^r W_r},$$

and, using (3.6), we deduce from this

$$AU_{2k+1}T_{2k+1} - AU_1 T_1 = -\sum_{r=1}^{2k+1} \frac{1}{\alpha^r W_r} = -A \sum_{r=1}^{k} \frac{1}{W_{2r} W_{2r+1}}.$$

This concludes the proof of (2.5) when $P > 0$. The case in which $P < 0$ is similar.

Using (4.5) and (4.6) and applying Corollary 1 to the sequence $\{W_n'\}$, one can easily obtain the proof of Corollary 2 when noticing that $Q^s = -1$, since s is odd.

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