BINOMIAL GRAPHS AND THEIR SPECTRA

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1. INTRODUCTION

Pascal's triangle with entries reduced modulo 2 has been the object of a variety of investigations, including number theoretical questions on the parity of binomial coefficients [4] and geometrical explorations of the self-similarity of the Sierpinski triangle [7]. Graph theory has also entered the scene as a consequence of various binary (that is, $\{0, 1\}$) matrix constructions that exploit properties of Pascal's triangle. For example, in [2] reference is made to *Pascal graphs* of order n whose (symmetric) adjacency matrix has zero diagonal and the first $n-1$ rows of Pascal's triangle, modulo 2, in the off diagonal elements. Constructions such as these are of special interest when the corresponding graphs unexpectedly reveal or reflect properties intrinsic to Pascal's triangle.

This is the case with *binomial graphs*, the subject of this paper. The adjacency matrices of these graphs are also related to Pascal's triangle, modulo 2. The graphs are found to exhibit a number of interesting properties including a graph property that relates to the Fibonacci sequence. Recall that the n^{th} Fibonacci number F_n appears in Pascal's triangle as the sum:

$$F_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k}.$$

Other properties of binomial graphs relate to the golden mean, to the Lucas numbers, and to several other features associated with Pascal's triangle.

2. BINOMIAL GRAPHS

For each nonnegative integer n, we define the *binomial graph* B_n to have vertex set $V_n = \{v_j : j = 0, 1, \dots, 2^n - 1\}$ and edge set $E_n = \{\{v_i, v_j\} : \binom{i+j}{j} \equiv 1 \pmod 2\}$. We define $\binom{0}{0} = 1$; thus, each binomial graph has a loop at v_0, but is otherwise a simple graph (that is, has no other loop and no multiedge). The binomial graph B_3 and its adjacency matrix $A(B_3)$ are depicted in Figure 1.

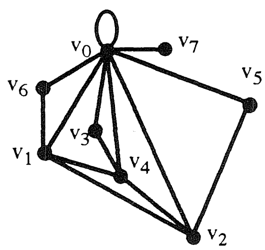
Obviously, $|V_n| = 2^n$. Also, for each $k = 0, 1, \dots, n-1$, B_n has $\binom{n}{k}$ vertices of degree 2^k and a single vertex, v_0, of degree $2^n + 1$. Thus, the sum of the degrees of vertices in B_n is

$$\sum_{k=0}^{n-1} \binom{n}{k} 2^k + (2^n + 1) = 1 + \sum_{k=0}^{n} \binom{n}{k} 2^k = 3^n + 1.$$

Consequently, $|E_n| = \frac{1}{2}(3^n + 1)$.

The adjacency matrix $A(B_n)$ exhibits a self-similarity. In this form, it can be described in terms of a Kronecker product of matrices. Recall that if $A=[a_{ij}]$ is an $m\times n$ matrix and $\boldsymbol{B}$ is a $p\times q$ matrix, then the Kronecker product $\boldsymbol{A}\otimes\boldsymbol{B}$ is the $mp\times nq$ matrix, $\boldsymbol{A}\otimes\boldsymbol{B}=[a_{ij}\boldsymbol{B}]$.

B_3



$A(B_3)$

	j = 0	1	2	3	4	5	6	7
i = 0	1	1	1	1	1	1	1	1
1	1	0	1	0	1	0	1	0
2	1	1	0	0	1	1	0	0
3	1	0	0	0	1	0	0	0
4	1	1	1	1	0	0	0	0
5	1	0	1	0	0	0	0	0
6	1	1	0	0	0	0	0	0
7	1	0	0	0	0	0	0	0

FIGURE 1. The Binomial Graph B_3 and Its Adjacency Matrix

Thus, if we take $A(B_0)=[1]$, then, for each $n\geq 1$, the adjacency matrix of the binomial graph B_n is

$$A(B_n)=\begin{bmatrix} A(B_{n-1}) & A(B_{n-1}) \\ A(B_{n-1}) & \mathbf{0} \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\otimes A(B_{n-1})=A(B_1)\otimes A(B_{n-1}).$$

3. SPECTRA OF BINOMIAL GRAPHS

The *eigenvalues of a graph* G are the eigenvalues of $A(G)$, the adjacency matrix of G. The *spectrum of a graph* is the sequence (or multiset) of its eigenvalues. We denote the spectrum of graph G by $\Lambda(G)$.

To obtain the spectrum of the binomial graph B_n, we exploit the following result concerning Kronecker products.

Lemma 1 (see [1]): Let $\boldsymbol{A}$ be an $n\times n$ matrix with (not necessarily distinct) eigenvalues $\lambda_1,\lambda_2,\ldots,\lambda_n$ and eigenvalues $x_1,x_2,\ldots,x_n$. Let $\boldsymbol{B}$ be an $m\times m$ matrix with eigenvalues $\mu_1,\mu_2,\ldots,\mu_m$ and eigenvectors $y_1,y_2,\ldots,y_m$. Then the Kronecker product $\boldsymbol{A}\otimes\boldsymbol{B}$ has nm eigenvalues $\lambda_i\mu_j$ and eigenvectors $x_i\otimes y_j$ for each $i=1,2,\ldots,n$ and each $j=1,2,\ldots,m$. $\square$

We use this lemma to establish that the eigenvalues of binomial graphs are powers of the golden mean, as are the entries in the corresponding eigenvectors.

Theorem 1: Let $\varphi=\frac{1}{2}(1-\sqrt{5})$. For each $n\geq 0$, the binomial graph B_n has $n+1$ distinct eigenvalues, specifically, $(-1)^j\varphi^{n-2j}$, for each $j=0,1,\ldots,n$. Each of these eigenvalues occurs with multiplicity $\binom{n}{j}$, so that the spectrum of B_n is

$$\Lambda(B\ )=[((-1)^j\varphi^{n-2j})^{\binom{n}{j}}: j=0,1,\ldots,n],$$

where $\lambda^{(m)}$ means that the eigenvalue λ has multiplicity m. Furthermore, for $n \geq 1$, 2^n linearly independent eigenvectors of B_n are scalar multiples of the columns in the Kronecker product $X(B_n) = X(B_1) \otimes X(B_{n-1})$, where $X(B_n) = [x_1, x_2, \ldots, x_{2^n}]$ is the matrix of eigenvectors of B_n with

$$X(B_0) = [1], \quad X(B_1) = \begin{bmatrix} 1 & 1 \\ 1/\varphi & -\varphi \end{bmatrix}.$$

Finally, the characteristic polynomial of B_n is

$$\mathcal{P}(B_n; x) = \prod_{j=0}^{n} [x - (-1)^j \varphi^{n-2j}]^{\binom{n}{j}}.$$

Proof: Since $A(B_0) = [1]$, obviously $\Lambda(B_0) = [1]$ and $\mathcal{P}(B_0; x) = x - 1$. Since

$$A(B_1) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$

then

$$\mathcal{P}(B_1; x) = \det \begin{bmatrix} x-1 & -1 \\ -1 & x \end{bmatrix} = x^2 - x - 1,$$

so that $\Lambda(B_1) = [\varphi, -\frac{1}{\varphi}]$, as required by the theorem. Furthermore, the two eigenvectors are $x_1^{\mathrm{T}} = [1, \varphi^{-1}]$ and $x_2^{\mathrm{T}} = [1, -\varphi]$ (or scalar multiples thereof), so that

$$X(B_1) = \begin{bmatrix} 1 & 1 \\ 1/\varphi & -\varphi \end{bmatrix}.$$

Since, for each $n > 1$

$$A(B_n) = A(B_1) \otimes A(B_{n-1}) = \underbrace{A(B_1) \otimes A(B_1) \otimes \cdots \otimes A(B_1)}_{n \text{ factors}},$$

then, by Lemma 1, the spectrum $\Lambda(B_n)$ consists of the n-fold (Cartesian) product of eigenvalues from the spectrum $\Lambda(B_1) = [\varphi, -\frac{1}{\varphi}]$. That is, the j^{th} distinct eigenvalue λ_j of B_n is the coefficient of $\binom{n}{j} t^j$ in the expansion of

$$\left(\varphi - \frac{t}{\varphi}\right)^n = \Sigma (-1)^j \binom{n}{j} \varphi^{n-2j} t^j,$$

and the multiplicity of λ_j is $\binom{n}{j}$. Furthermore, also by Lemma 1, $X(B_n) = X(B_1) \otimes X(B_{n-1})$. □

4. CHARACTERISTIC POLYNOMIALS OF BINOMIAL GRAPHS

A polynomial of degree n, $P(x) = \sum_{k=0}^{n} c_k x^k$, $c_0 \neq 0$, is called *palindromic* if, for each $k = 0, \ldots, n$, $|c_k| = |c_{n-k}|$ (see [3] and [6]). Some interest attaches to graphs whose characteristic polynomials are palindromic. A palindromic polynomial is said to be *exactly palindromic* if, for each k, $c_k = c_{n-k}$ and *skew palindromic* if $c_k = -c_{n-k}$. A palindromic polynomial of even degree is called *even pseudo palindromic* if, for each k, $c_k = (-1)^k c_{n-k}$ and *odd pseudo palindromic* if $c_k = -(-1)^k c_{n-k}$.

By expressing the characteristic polynomials of binomial graphs as products of simple (unit) quadratic factors involving the Lucas numbers, we show that the binomial graphs are palindromic with respect to their characteristic polynomials.

From Theorem 1, $\mathcal{P}(B_0; x) = x - 1$ is obviously skew palindromic. For even $n > 0$,

$$\begin{aligned}\mathcal{P}(B_n; x) &= \prod_{j=0}^{n}[x - (-1)^j \varphi^{n-2j}]^{\binom{n}{j}} \\ &= (x - (-1)^{n/2})^{\binom{n}{n/2}} \prod_{j=0}^{(n-2)/2} [x^2 - (-1)^j(\varphi^{n-2j} + \hat{\varphi}^{n-2j})x + (-1)^n]^{\binom{n}{j}},\end{aligned}$$

where $\hat{\varphi} = -\frac{1}{\varphi}\left(= \frac{1-\sqrt{5}}{2}\right)$. Since, for even $n > 0$, the central binomial coefficient $\binom{n}{n/2}$ is even, then

$$\mathcal{P}(B_n; x) = (x^2 - (-1)^{n/2} L_0 x + 1)^{\frac{1}{2}\binom{n}{n/2}} \prod_{j=0}^{(n-2)/2} [x^2 - (-1)^j L_{n-2j} x + 1]^{\binom{n}{j}},$$

where L_k is the k^{th} Lucas number for $k \geq 1$ and $L_0 = 2$. Consequently, for even $n > 0$, $\mathcal{P}(B_n; x)$ is a product of exact palindromic (quadratic) polynomials; hence, see Lemma 2.2 in [3], $\mathcal{P}(B_n; x)$ is exact palindromic.

For n odd, B_n has no eigenvalue of unit magnitude, but similarly,

$$\mathcal{P}(B_n; x) = \prod_{j=0}^{(n-1)/2} [x^2 - (-1)^j L_{n-2j} x - 1]^{\binom{n}{j}},$$

so that $\mathcal{P}(B_n; x)$ is a product of 2^{n-1} odd pseudo palindromic polynomials. Obviously $\mathcal{P}(B_n; x)$ is odd pseudo palindromic but (see [3], Lemma 2.2), for each odd $n > 1$, $\mathcal{P}(B_n; x)$ is even pseudo palindromic.

Note that for each binomial graph B_n with $n > 1$, the characteristic polynomial $\mathcal{P}(B_n; x)$ can be expressed as a product of unit quadratic factors whose central coefficients are Lucas numbers L_k with $k \equiv n \pmod 2$.

5. CLOSED WALKS IN BINOMIAL GRAPHS

As was observed by P. W. Kasteleyn [5], the characteristic polynomial $\mathcal{P}(G; x)$ of a graph G can be applied to determine the number of closed walks of fixed length in G. We state this result as

Lemma 2: The total number of closed walks of length k in a graph G is the coefficient of t^k in the generating function

$$W(G; t) = \frac{\mathcal{P}'(G; \frac{1}{t})}{t\mathcal{P}(G; \frac{1}{t})}, \text{ where } \mathcal{P}'(G; x) = \frac{d}{dx}\mathcal{P}(G; x). \quad \square$$

By applying this lemma to the graphs B_n, we obtain a connection between binomial graphs and the Lucas numbers.

Theorem 2: The (ordinary) generating function for the total number of closed walks of length k in the binomial graph B_n is

$$W(B_n;t)=\sum_{k=0}^{\infty} L_k^n t^k,$$

where L_k is the k^{th} Lucas number for $k \ge 1$ and $L_0 = 2$.

Proof: By Lemma 2,

$$W(B_n;t)=\frac{\mathcal{P}'\left(B_n;\frac{1}{t}\right)}{t\mathcal{P}\left(B_n;\frac{1}{t}\right)},$$

where, from Theorem 1,

$$\mathcal{P}(B_n;x)=\prod_{j=0}^{n}[x-(-1)^j\varphi^{n-2j}]^{\binom{n}{j}}.$$

Setting $\hat{\varphi}=-\frac{1}{\varphi}\left(=\frac{1-\sqrt{5}}{2}\right)$, we can write

$$\mathcal{P}(B_n;x)=\prod_{j=0}^{n}(x-\hat{\varphi}^j\varphi^{n-j})^{\binom{n}{j}}.$$

Taking the logarithm of both sides and differentiating with respect to x yields

$$\frac{\mathcal{P}'(B_n;x)}{\mathcal{P}(B_n;x)}=\sum_{j=0}^{n}\frac{\binom{n}{j}}{x-\hat{\varphi}^j\varphi^{n-j}}.$$

It follows that

$$W(B_n;t)=\sum_{j=0}^{n}\frac{\binom{n}{j}}{1-\hat{\varphi}^j\varphi^{n-j}t}=\sum_{j=0}^{n}\binom{n}{j}\sum_{k=0}^{\infty}\hat{\varphi}^{jk}\varphi^{(n-j)k}t^k$$

$$=\sum_{k=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j}\hat{\varphi}^{jk}\varphi^{(n-j)k}\right)t^k=\sum_{k=0}^{\infty}(\varphi^k+\hat{\varphi}^k)^n t^k=\sum_{k=0}^{\infty}L_k^n t^k. \quad \square$$

Consider now the number of closed walks of length k in B_n with initial (and final) vertex v_0. Let $W_0(B_n;t)$ denote the generating function for this sequence. To determine the coefficients of this generating function, we first need the following lemma.

Lemma 3: Let $v_j \in V(B_n)$ with the vertices labeled in natural order $\{0,1,\ldots,2^n-1\}$ and let $w_n(j)$ denote the representation of the natural number j as a binary word of length n. Then $\{v_i, v_j\} \in E(B_n)$ if and only if $w_n(i)$ and $w_n(j)$ have no 1-bit in common.

Proof: The lemma is an immediate consequence of the fact that

$$\binom{i+j}{i}=\binom{i+j}{j}\equiv 1 \pmod 2$$

if and only if $w_n(i)$ and $w_n(j)$ have no 1-bit in common. $\square$

Theorem 3: The number of closed walks of length k with initial vertex v_0 in B_n is the coefficient of t^k in the generating function

$$W_0(B_n;t)=\sum_{k=0}^{\infty}F_{k+1}^n t^k$$

where F_k is the k^{th} Fibonacci number.

Proof: The statement is easily verified for $k = 0$ or 1: the number of closed walks starting at v_0 in B_n is equal to 1 in each case. For $k \geq 2$, a walk of length k in B_n can be described as an ordered list of $k+1$ vertices. Let each vertex v_j $(j = 0, 1, \ldots, 2^n - 1)$ be labeled with the corresponding binary word, $w_n(j)$, of length n. Then a walk of length k in B_n can be described as an ordered list of $k+1$ binary words each of length n and such that no two consecutive words have a 1-bit in common. Obviously, for a closed walk commencing at vertex v_0, the first and last binary word is the zero word $w_n(0)$.

Consider the $(k-1) \times n$ matrix $\boldsymbol{M}$, whose *rows* in sequence are the binary words describing a closed walk in B_n starting at v_0, with the first and last word $w_n(0)$ deleted. Now the *columns* of $\boldsymbol{M}$ can be viewed as n independent and ordered $\{0, 1\}$-sequences of length $k-1$, with the property that no two 1-bits are adjacent. Since there are exactly F_{k+1} such sequences, where F_k is the k^{th} Fibonacci number, it follows that there are F_{k+1}^n binary words of length n in which no two consecutive words have a 1-bit in common. That is, the number of closed walks of length k from v_0 in B_n is F_{k+1}^n. □

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