

ON A RECURRENCE RELATION IN TWO VARIABLES

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1. INTRODUCTION

Gupta [3] considers the array $\{c(n, k)\}$, which is defined by the recurrence relation

$$c(n+1, k) = c(n, k) + c(n, k-1) \quad (1)$$

with initial values $c(n, 0) = a(n)$, $c(1, k) = b(k)$, $n, k \geq 1$, where $\{a(n)\}$ and $\{b(k)\}$ are given sequences of numbers. In particular, if $a(n) \equiv 1$ and $b(1) = 1$, $b(k) = 0$ for $k \geq 2$, then $\{c(n, k)\}$ is the classical binomial array. The array $\{c(n, k)\}$ also has applications, for example, in the theory of partitions of integers ([1], [2]). The main object of Gupta [3] is to handle $\{c(n, k)\}$ with the aid of generating functions. Wilf [7, §§1.5 and 1.6] handles $\{c(n, k)\}$ and an analog of $\{c(n, k)\}$, namely, the array of the Stirling numbers of the second kind, with the aid of generating functions.

In this paper we consider a further analog of the array $\{c(n, k)\}$, namely, the array $\{L(n, k)\}$ defined by the recurrence relation

$$L(n, k) = L(n-1, k-1) - L(n, k-1) \quad (2)$$

with initial values $L(n, 0) = a(n)$, $L(0, k) = b(k)$, $L(0, 0) = a(0) = b(0)$, $n, k \geq 1$, where $\{a(n)\}$ and $\{b(k)\}$ are given sequences of numbers. We derive an expression for the numbers $L(n, k)$ in terms of the initial values using the method of generating functions. We motivate the study of the array $\{L(n, k)\}$ by providing a concrete example of this kind of array from the theory of stack filters.

2. AN EXPRESSION FOR THE NUMBERS $L(n, k)$

Let $L_k(x)$ denote the generating function of the sequence $\{L(n, k)\}_{n=0}^{\infty}$, that is,

$$L_k(x) = \sum_{n=0}^{\infty} L(n, k)x^n. \quad (3)$$

The promised expression for the numbers $L(n, k)$ comes out as follows: We use the recurrence relation (2) to obtain a recurrence relation for the generating function $L_k(x)$. This recurrence relation yields an expression for the generating function $L_k(x)$, which gives an expression for the numbers $L(n, k)$.

Theorem 1: Let $\{L(n, k)\}$ be the array given in (2). Then

$$L(n, k) = \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} a(n-j) + \sum_{j=n}^{k-1} \binom{j}{n} (-1)^{j-n} [b(k-j) - b(k-j-1)]. \quad (4)$$

Proof: If $k = 0$, then (4) holds. Let $k \geq 1$. Then, by (3) and (4),

$$\begin{aligned} L_k(x) &= L(0, k) + \sum_{n=1}^{\infty} [L(n-1, k-1) - L(n, k-1)]x^n \\ &= L(0, k) + [xL_{k-1}(x) - (L_{k-1}(x) - L(0, k-1))] \end{aligned}$$

or

$$L_k(x) = (x-1)L_{k-1}(x) + b(k) + b(k-1).$$

Let $d(k) = b(k) + b(k-1)$. Proceeding by induction on k , we obtain

$$L_k(x) = (x-1)^k L_0(x) + \sum_{j=0}^{k-1} (x-1)^j d(k-j). \quad (5)$$

Here

$$\begin{aligned} (x-1)^k L_0(x) &= \left(\sum_{n=0}^{\infty} \binom{k}{n} (-1)^{k-n} x^n \right) \left(\sum_{n=0}^{\infty} a(n) x^n \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{j=0}^n \binom{k}{j} (-1)^{k-j} a(n-j) \right) x^n \end{aligned} \quad (6)$$

and

$$\begin{aligned} \sum_{j=0}^{k-1} (x-1)^j d(k-j) &= \sum_{j=0}^{k-1} \sum_{n=0}^j \binom{j}{n} x^n (-1)^{j-n} d(k-j) \\ &= \sum_{n=0}^{k-1} \left(\sum_{j=n}^{k-1} \binom{j}{n} (-1)^{j-n} d(k-j) \right) x^n. \end{aligned} \quad (7)$$

Now, combining (3), (5), (6), and (7), we obtain (4).

Remark: We considered above the generating function of the array $\{L(n, k)\}$ with respect to the variable n . We could also consider generating functions with respect to the variable k and with respect to both the variables n and k . These considerations, however, would be more laborious and are not presented here (cf. [3], [7]).

3. AN EXAMPLE FROM THE THEORY OF STACK FILTERS

Consider a stack filter (for definition, see [4], [6]) with continuous i.i.d. inputs having distribution function $\Phi(\cdot)$ and with window size N . The γ -order moment about the origin of the output can be written as

$$\alpha^\gamma = E\{Y_{\text{out}}^\gamma\} = \sum_{k=0}^{N-1} A_k M(\Phi, \gamma, N, k),$$

where

$$M(\Phi, \gamma, N, k) = \int_{-\infty}^{\infty} x^\gamma \frac{d}{dx} \left((1 - \Phi(x))^k \Phi(x)^{N-k} \right) dx, \quad k = 0, 1, \dots, N-1, \quad (8)$$

and where the coefficients A_k , $k = 0, 1, \dots, N-1$, have a certain natural interpretation (see [4], [5]). By using the output moments about the origin, we easily obtain output central moments, denoted by $\mu^\gamma = E\{(Y_{\text{out}} - E\{Y_{\text{out}}\})^\gamma\}$, for example, the second-order central output moment equals

$$\mu^2 = \sum_{k=0}^{N-1} A_k M(\Phi, 2, N, k) - \left(\sum_{k=0}^{N-1} A_k M(\Phi, 1, N, k) \right)^2. \quad (9)$$

The second-order central output moment is quite often used as a measure of the noise attenuation capability of a filter. It quantifies the spread of the input samples with respect to their mean value. Equation (9) gives an expression for the second-order central output moment. In this expression, the numbers $M(\Phi, \gamma, N, k)$ play a crucial role.

Kuosmanen [4] and Kuosmanen & Astola [5] studied the properties of the numbers $M(\Phi, \gamma, N, k)$ under certain conditions on $\Phi(\cdot)$. They showed, among other things, that the numbers $M(\Phi, \gamma, N, k)$ satisfy the recurrence relation

$$M(\Phi, \gamma, N, k) = M(\Phi, \gamma, N-1, k-1) - M(\Phi, \gamma, N, k-1), \quad 1 \leq k \leq N,$$

with initial values

$$M(\Phi, \gamma, N, 0) = \int_{-\infty}^{\infty} x^\gamma \frac{d}{dx} (\Phi(x)^N) dx, \quad N \geq 0.$$

This means that the numbers $M(\Phi, \gamma, N, k)$ satisfy recurrence relation (2). As $M(\Phi, \gamma, N, k) = 0$ if $N = 0$, that is, as $b(k) \equiv 0$ in (2), application of Theorem 1 gives

$$M(\Phi, \gamma, N, k) = \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} M(\Phi, \gamma, N-j, 0). \quad (10)$$

Note that Kuosmanen [4] and Kuosmanen & Astola [5] derived (10) directly from (8) using the binomial theorem.

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