THE CONSTANT FOR FINITE DIOPHANTINE APPROXIMATION

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Let x be an irrational number. In 1891, Hurwitz [3] proved that there are infinitely many rational numbers p/q such that p and q are coprime integers and $|x-p/q|<1/(\sqrt{5}q^2)$. Hurwitz' theorem has been extensively investigated (see [6]).

In 1948, following Davenport's suggestion, Prasad [4] initiated the study of finite Diophantine approximation. He proved that, for any given irrational number x, and any given positive integer m, there is a constant C_m such that the inequality $|x-p/q|<1/(C_m q^2)$ has at least m rational solutions p/q. In [4], the structure of C_m has been mentioned, and $C_1=(3+\sqrt{5})/2$ has been calculated, but the values of C_m as a function of m is still unknown.

In this note we will use the Fibonacci sequence to prove that

$$C_m=\sqrt{5}+\frac{\sqrt{5}}{\left(\frac{7+3\sqrt{5}}{2}\right)^m-1}. \tag{1}$$

Theorem 1: Let x be an irrational number. If m is a given positive integer, then there are at least m rational numbers p/q such that p and q are coprime integers and $|x-p/q|<1/(C_m q^2)$, where C_m is as shown in formula (1). The constants C_m cannot be replaced by a smaller number.

Proof: Let $x=[a_0; a_1, a_2, \ldots, a_n, \ldots]$ be the expansion of x in a simple continued fraction. Let $p_n/q_n=[a_0; a_1, \ldots, a_n]$ be the n^{th} convergent, then p_n and q_n are coprime integers. It is well known that (see [5])

$$|x-p_n/q_n|=1/(M_n q_n^2),$$

where $M_n=a_{n+1}+[0; a_{n+2}, a_{n+3}, \ldots]+[0; a_n, a_{n-1}, \ldots, a_1]$.

By Legendre's theorem [5], $|x-p/q|<1/(2q^2)$ implies that p/q must be a convergent p_n/q_n for some n. Thus, we need only discuss the rational solutions of $|x-p/q|<1/(C_m q^2)$ among the convergents p_n/q_n.

We discuss the following possible cases on the partial quotients a_n. It is easily seen that $C_m\le C_1<8/3<3$.

Suppose there are infinitely many $a_n\ge 3$, then $M_{n-1}\ge a_n\ge 3\ge C_m$ for all positive integer m. Hence, we need only consider the case in which there are only finitely many $a_n\ge 3$. That is to say, there is a positive integer N_1 such that $n\ge N_1$ implies $a_n\le 2$. We consider two cases.

Case 1. There are infinitely many a_n such that $a_n=2$. Then, for these n, $n>N_1+2$ implies $M_{n-1}\ge 2+[0; 2, 1]+[0; 2, 1]=8/3>C_m$ for all positive integers m.

Case 2. There are finitely many $a_n=2$. Thus, there is a positive integer $N_2\ge N_1$ such that $n\ge N_2$ implies $a_n=1$.

Let $N = \max\{n, a_n \neq 1\}$. Then $a_N \geq 2$, $a_{N+1} = a_{N+2} = \cdots = 1$. Therefore, if we use $[0; (1)_k]$ to denote $[0; 1, \ldots, 1]$ with k consecutive 1's, the following inequalities are true because $a_{N+1} = a_{N+2} = \cdots = a_{N+2m-1} = 1$; there are $2m-1$ consecutive 1's.

$$\begin{aligned} M_{N+2m-1} &= a_{N+2m} + [0; \overline{1}] + [0; 1, \ldots, 1, a_N, a_{N-1}, \ldots, a_1] \\ &\geq 1 + [0; \overline{1}] + [0; (1)_{2m-1}]. \end{aligned} \tag{2}$$

Similarly, we have

$$\begin{aligned} M_{N+2m+1} &= a_{N+2m+2} + [0; \overline{1}] + [0; 1, \ldots, 1, a_N, a_{N-1}, \ldots, a_1] \\ &\geq 1 + [0; \overline{1}] + [0; (1)_{2m+1}], \end{aligned}$$

$$\ldots,$$

$$\begin{aligned} M_{N+4m-3} &= a_{N+4m-2} + [0; \overline{1}] + [0; 1, \ldots, 1, a_N, a_{N-1}, \ldots, a_1] \\ &\geq 1 + [0; \overline{1}] + [0; (1)_{4m-3}]. \end{aligned}$$

It is easily seen that $M_{N+2m-1} < M_{N+2m+1} < \cdots < M_{N+4m-3}$. Denoting $C_m = M_{N+2m-1}$, then the inequality $|x - p/q| < 1/(C_m q^2)$ has at least m rational solutions p_n / q_n.

Now we calculate C_m with the help of the Fibonacci sequence.

Let $F_1 = 1, F_2 = 1, F_n = F_{n-1} + F_{n-2}$ be the Fibonacci sequence. We are going to find a formula for $[0; (1)_{2m-1}]$ by mathematical induction.

It is easily seen that $[0; (1)_1] = [0; 1] = 1/1 = F_1 / F_2$. Suppose $[0; (1)_{2k-1}] = F_{2k-1} / F_{2k}$, then we have $[0; (1)_{2(k+1)-1}] = [0; 1, 1, (1)_{2k-1}] = 1/(1 + (1 + F_{2k-1}/F_{2k})) = F_{2k+1}/F_{2k+2}$. Thus, $[0; (1)_{2m-1}] = F_{2m-1} / F_{2m}$.

By Binet's formula for the Fibonacci sequence [1], i.e., $F_n = ((1+\sqrt{5})^n - (1-\sqrt{5})^n)/(2^n\sqrt{5})$, we can find F_{2m-1}/F_{2m} as follows:

$$\begin{aligned} \frac{F_{2m-1}}{F_{2m}} &= \frac{2((1+\sqrt{5})^{2m-1} - (1-\sqrt{5})^{2m-1})}{(1+\sqrt{5})^{2m} - (1-\sqrt{5})^{2m}} = \frac{\sqrt{5}((1+\sqrt{5})^{2m} + (1-\sqrt{5})^{2m})}{2((1+\sqrt{5})^{2m} - (1-\sqrt{5})^{2m})} - \frac{1}{2} \\ &= \frac{\sqrt{5}(1 + (\sqrt{5}-3)/2)^{2m}}{2(1 - (\sqrt{5}-3)/2)^{2m}} - \frac{1}{2} = \frac{\sqrt{5}}{2}\left(1 + \frac{2((\sqrt{5}-3)/2)^{2m}}{1 - ((\sqrt{5}-3)/2)^{2m}}\right) - \frac{1}{2} \\ &= \frac{\sqrt{5}}{2}\left(1 + \frac{2}{((3+\sqrt{5})/2)^{2m} - 1}\right) - \frac{1}{2}. \end{aligned}$$

Notice that because $[0; \overline{1}] = (\sqrt{5}-1)/2$ we have, by formula (2), that

$$C_m = M_{N+2m-1} = 1 + (\sqrt{5}-1)/2 + F_{2m-1}/F_{2m},$$

which gives formula (1).

The constants C_m cannot be replaced by smaller numbers since, for $x = [0; \overline{1}]$, we have exactly $C_m = M_{2m-1} = 1 + [0; \overline{1}] + [0; (1)_{2m-1}]$. □

Corollary 1: $C_1 = (3+\sqrt{5})/2 = 2.6180$,
$C_2 = (7+3\sqrt{5})/6 = 2.2847$,
$C_3 = (9+4\sqrt{5})/8 = 2.2430$.

Corollary 2: $\lim_{m\to\infty} C_m = \sqrt{5} = 2.2361$.

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