ADVANCED PROBLEMS AND SOLUTIONS

Edited by Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-526 ***Proposed by Paul S. Bruckman, Highwood, IL***

Following H-465, let r_1, r_2, and r_3 be natural integers such that

$$\sum_{k=1}^{3} kr_k = n, \text{ where } n \text{ is a given natural integer.} \tag{1}$$

Let

$$B_{r_1,r_2,r_3} = \frac{1}{r_1+r_2+r_3}\frac{(r_1+r_2+r_3)!}{r_1!r_2!r_3!}. \tag{2}$$

Also, let

$$C_n = \sum B_{r_1,r_2,r_3}, \text{ summed over all possible } r_1, r_2, r_3. \tag{3}$$

Define the generating function

$$F(x) = \sum_{n=6}^{\infty} C_n x^n. \tag{4}$$

(a) Find a closed form for $F(x)$;

(b) Obtain an explicit expression for C_n;

(c) Show that C_n is a positive integer for all $n \geq 7$, n prime.

H-527 ***Proposed by N. Gauthier, Royal Military College of Canada***

Let q, a, and b be positive integers, with $(a, b) = 1$. Prove or disprove the following:

a) $$\sum_{\substack{r=0 \\ (br+as<ab)}}^{a-1} \sum_{s=0}^{b-1} (-1)^{q(br+as)} L_{2q(br+as)} = \frac{F_{q(a+b-ab)}F_{qab}}{F_{qa}F_{qb}} + (-1)^{q(1-ab)}\frac{F_{q(2ab-1)}}{F_q};$$

b) $$5\sum_{\substack{r=0 \\ (br+as<ab)}}^{a-1} \sum_{s=0}^{b-1} (-1)^{q(br+as)} F_{2q(br+as)} = (-1)^{q(1-ab)}\frac{L_{q(2ab-1)}}{F_q} - \frac{F_{qab}L_{q(a+b-ab)}}{F_{qa}F_{qb}}.$$

H-528 ***Proposed by Paul S. Bruckman, Highwood, IL***

Let $\Omega(n) = \sum_{p^e \| n} e$, given the prime decomposition of a natural number $n = \prod p^e$. Prove the following:

(A) $$\sum_{d|n}(-1)^{\Omega(d)}F_{\Omega(n/d)-\Omega(d)}=0;$$

(B) $$\sum_{d|n}(-1)^{\Omega(d)}L_{\Omega(n/d)-\Omega(d)}=2U_n,\text{ where } U_n=\prod_{p^e\|n}F_{e+1}.$$

SOLUTIONS

Poly Forms

H-508 (Corrected) ***Proposed by H.-J. Seiffert, Berlin, Germany (Vol. 34, no. 1, February 1995)***

Define the Fibonacci polynomials by $F_0(x)=0$, $F_1(x)=1$, $F_n(x)=xF_{n-1}(x)+F_{n-2}(x)$, for $n\ge 2$. Show that, for all complex numbers x and y and all positive integers n,

$$F_n(x)F_n(y)=n\sum_{k=0}^{n-1}\frac{1}{k+1}\binom{n+k}{2k+1}(x+y)^kF_{k+1}\left(\frac{xy-4}{x+y}\right). \tag{1}$$

As special cases of (1), obtain the following identities:

$$F_n(x)F_n(x+1)=n\sum_{k=0}^{n-1}\frac{(-1)^{n-k+1}}{k+1}\binom{n+k}{2k+1}F_{k+1}(x^2+x+4); \tag{2}$$

$$F_n(x)F_n(4/x)=n\sum_{k=0}^{[\frac{n-1}{2}]}\frac{1}{2k+1}\binom{n+2k}{4k+1}\left(\frac{x^2+4}{x}\right)^{2k},\quad x\ne 0; \tag{3}$$

$$F_n(x)^2=n\sum_{k=0}^{n-1}\frac{(-1)^{n-k+1}}{k+1}\binom{n+k}{2k+1}(x^2+4)^k; \tag{4}$$

$$F_n(x)^2=n\sum_{k=0}^{n-1}\frac{1}{k+1}\binom{n+k}{2k+1}\frac{x^{2k+2}-(-4)^{k+1}}{x^2+4}; \tag{5}$$

$$F_{2n-1}(x)=(2n-1)\sum_{k=0}^{2n-2}\frac{(-1)^k}{k+1}\binom{2n+k-1}{2k+1}x^kF_{k+1}(4/x). \tag{6}$$

Solution by the proposer

We also consider the Lucas polynomials defined by $L_0(x)=2$, $L_1(x)=x$, $L_n(x)=xL_{n-1}(x)+L_{n-2}(x)$, for $n\ge 2$. It is known that

$$F_n(x)=\frac{\alpha(x)^n-\beta(x)^n}{\sqrt{x^2+4}}\quad\text{and}\quad L_n(x)=\alpha(x)^n+\beta(x)^n, \tag{7}$$

where $\alpha(x)=\frac{1}{2}\left(x+\sqrt{x^2+4}\right)$ and $\beta(x)=\frac{1}{2}\left(x-\sqrt{x^2+4}\right)$, and that

$$F_{2n}(x)=\sum_{k=0}^{n-1}\binom{n+k}{2k+1}x^{2k+1}.$$

Integrating the latter equation and noting that $L'_{2n}(x)=2nF_{2n}(x)$ and $L_{2n}(0)=2$ gives

$$L_{2n}(x)-2=n\sum_{k=0}^{n-1}\frac{1}{k+1}\binom{n+k}{2k+1}x^{2k+2}. \tag{8}$$

Since both sides of the stated equation (1) are analytic functions of x and y, it suffices to prove it for real x and y such that $x\ge y>0$. Let

$$u=\tfrac{1}{2}\left(\sqrt{(x^2+4)(y^2+4)}+xy-4\right)$$

and

$$v=\tfrac{1}{2}\left(\sqrt{(x^2+4)(y^2+4)}-xy+4\right).$$

Then we have $u>0$ and $v>4$. From (8) it follows that

$$\frac{L_{2n}(\sqrt{u})-L_{2n}(i\sqrt{v})}{u+v}=n\sum_{k=0}^{n-1}\frac{1}{k+1}\binom{n+k}{2k+1}A_{k+1}, \tag{9}$$

where $i=\sqrt{(-1)}$ and

$$A_j:=\frac{u^j-(i\sqrt{v})^{2j}}{u+v},\quad j\in\mathbb{N}_0.$$

Since $u-v=xy-4$ and $uv=(x+y)^2$, it is easily seen that

$$A_j=(xy-4)A_{j-1}+(x+y)^2A_{j-2},\quad j\ge 2,$$

so that, by $A_0=0$ and $A_1=1$, we must have

$$A_j=(x+y)^{j-1}F_j\left(\frac{xy-4}{x+y}\right),\ j\in\mathbb{N}_0. \tag{10}$$

Simple calculations show that

$$\alpha(\sqrt{u})^2=\tfrac{1}{4}\left(2u+4+2\sqrt{u(u+4)}\right)=\alpha(x)\alpha(y),$$
$$\beta(\sqrt{u})^2=\tfrac{1}{4}\left(2u+4-2\sqrt{u(u+4)}\right)=\beta(x)\beta(y),$$

and, since $x\ge y$,

$$\alpha(i\sqrt{v})^2=-\tfrac{1}{4}\left(2v-4+2\sqrt{v(v-4)}\right)=\alpha(x)\beta(y),$$
$$\beta(i\sqrt{v})^2=-\tfrac{1}{4}\left(2v-4-2\sqrt{v(v-4)}\right)=\beta(x)\alpha(y).$$

From these four equations and (7), it follows that

$$F_n(x)F_n(y)=\frac{L_{2n}(\sqrt{u})-L_{2n}(i\sqrt{v})}{u+v}. \tag{11}$$

Now, the desired identity (1) follows from (9), (10), and (11).

Using the properties $F_j(-x)=(-1)^{j-1}F_j(x)$, $F_{2j}(0)=0$, and $F_{2j+1}(0)=1$, we show that (2)-(6) are all special cases of (1). Since we wish to exhibit some particular cases, we also note that $F_n(4)=F_{3n}/2$, $F_n(3i)=i^{n-1}F_{2n}$, $F_{2n}(\sqrt{5})=\sqrt{5}F_{4n}/3$, $F_{2n-1}(\sqrt{5})=L_{4n-2}/3$, and $5^{(n-1)/2}F_n(4/\sqrt{5})=(5^n-(-1)^n)/6$. Also, let $P_n=F_n(2)$ denote the n^{th} Pell number.

(2): In (1), replace x by $-x$ and then take $y=x+1$. We note the interesting particular case

$$F_nP_n = n\sum_{k=0}^{n-1}\frac{(-1)^{n-k+1}}{k+1}\binom{n+k}{2k+1}F_{k+1}(6).$$

(3): Take $y = 4/x, x \neq 0$. For $x=1$ and $x=2$, we obtain, respectively,

$$F_nF_{3n} = 2n\sum_{k=0}^{[\frac{n-1}{2}]}\frac{1}{2k+1}\binom{n+2k}{4k+1}25^k, \text{ and } P_n^2 = n\sum_{k=0}^{[\frac{n-1}{2}]}\frac{1}{2k+1}\binom{n+2k}{4k+1}16^k.$$

With $x=\sqrt{5}$, eq. (3), after replacing n by $2n$ and n by $2n-1$, produces the curious identities:

$$F_{4n} = \frac{36n}{25^n-1}\sum_{k=0}^{n-1}\frac{1}{2k+1}\binom{2n+2k}{4k+1}81^k5^{n-1-k};$$

$$L_{4n-2} = \frac{18(2n-1)}{5^{2n-1}+1}\sum_{k=0}^{n-1}\frac{1}{2k+1}\binom{2n+2k-1}{4k+1}81^k5^{n-1-k}.$$

(4): Take $y=-x$. For $x=1, x=3i$, and $x=2$, we obtain, respectively,

$$F_n^2 = n\sum_{k=0}^{n-1}\frac{(-1)^{n-k+1}}{k+1}\binom{n+k}{2k+1}5^k,$$

$$F_{2n}^2 = n\sum_{k=0}^{n-1}\frac{1}{k+1}\binom{n+k}{2k+1}5^k,$$

and

$$P_n^2 = n\sum_{k=0}^{n-1}\frac{(-1)^{n-k+1}}{k+1}\binom{n+k}{2k+1}8^k.$$

(5): Take $y=x$ and use the Binet form of the Fibonacci polynomials. For $x=3i$, this gives

$$F_{2n}^2 = \frac{n}{5}\sum_{k=0}^{n-1}\frac{(-1)^{n-k+1}}{k+1}\binom{n+k}{2k+1}(9^{k+1}-4^{k+1}).$$

(6): In (1), replace n by $2n-1$ and then set $y=0$. For $x=1$, $x=4$, $x=\sqrt{5}$, and $x=2$, we obtain, respectively,

$$F_{2n-1} = \frac{2n-1}{2}\sum_{k=0}^{2n-2}\frac{(-1)^k}{k+1}\binom{2n+k-1}{2k+1}F_{3k+3},$$

$$F_{6n-3} = (4n-2)\sum_{k=0}^{2n-2}\frac{(-1)^k}{k+1}\binom{2n+k-1}{2k+1}4^kF_{k+1},$$

$$L_{4n-2} = \frac{2n-1}{2}\sum_{k=0}^{2n-2}\frac{(-1)^k}{k+1}\binom{2n+k-1}{2k+1}(5^{k+1}-(-1)^{k+1}),$$

and

$$P_{2n-1} = (2n-1)\sum_{k=0}^{2n-2}\frac{(-1)^k}{k+1}\binom{2n+k-1}{2k+1}2^kP_{k+1}.$$

Also solved by P. Bruckman and A. Dujella.

Pell Mell

H-510 ***Proposed by H.-J. Seiffert, Berlin, Germany***
(Vol. 34, no. 2, May 1996)

Define the Pell numbers by $P_0 = 0, P_1 = 1, P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$. Show that

$$P_n = \sum_{k \in A_n} (-1)^{[(3k-2n-1)/4} 2^{[3k/2]} \binom{n+k}{2k+1}, \text{ for } n = 1, 2, \ldots,$$

where [ ] denotes the greatest integer function and $A_n = \{k \in \{0, 1, \ldots, n-1\} \mid 3k \not\equiv 2n \pmod 4\}$.

Solution by the proposer

First, we prove two theorems concerning the Fibonacci polynomials defined by

$$(1 - xz - z^2)^{-1} = \sum_{n=0}^{\infty} F_{n+1}(x) z^n, \tag{1}$$

which are also of interest in themselves.

Theorem 1: For all real x one has

$$F_{n+1}(x) = \sum_{k=0}^{n} \binom{n+k+1}{2k+1} i^{n-k} (x - 2i)^k, \text{ where } i^2 = -1.$$

Proof: Consider the special Jacobi polynomials defined by

$$(1 - 2xz + z^2)^{-j} = \sum_{n=0}^{\infty} C_n^j(x) z^n, \; j = 1, 2, \ldots. \tag{2}$$

It is well known [1, p. 374] that C_n^j has the derivatives

$$\frac{d^k}{dx^k} C_n^j(x) = 2^k \frac{(j+k-1)!}{(j-1)!} C_{n-k}^{j+k}(x). \tag{3}$$

If we substitute z by iz in (2) and compare the newly obtained equation with (1), we see that $F_{n+1}(x) = i^n C_n^1(x/2i)$. Thus, we have (3), and simple calculation gives

$$\frac{d^k}{dx^k} F_{n+1}(x) = k! \, i^{n-k} C_{n-k}^{k+1}(x/2i). \tag{4}$$

Since F_{n+1} is a polynomial of degree n, and since [1, p. 374]

$$C_{n-k}^{k+1}(1) = \binom{n+k+1}{2k+1},$$

the stated equation follows from (4) and Taylor's theorem. Q.E.D.

Theorem 2: For positive reals x one has

$$F_{n+1}(x) = \sum_{k=0}^{n} \binom{n+k+1}{2k+1} \Delta^k \cos \alpha_k$$

where $\Delta = (x^2 + 4)^{1/2}$ and $\alpha_k = (n-k)\pi/2 - k \arccos(x/\Delta)$.

Proof: Since $i = e^{i\pi/2}$ and $x - 2i = \Delta \exp(-i \arccos(x/\Delta))$, Theorem 1 gives

$$F_{n+1}(x)=\sum_{k=0}^{n}\binom{n+k+1}{2k+1}\Delta^k\exp(i\alpha_k),$$

which implies the stated equation by separating the real part. Q.E.D.

Now we are able to prove the proposer's equation. Using $P_{n+1}=F_{n+1}(2)$ and $\cos(\pi/4)=1/\sqrt{2}$, Theorem 2 gives

$$P_{n+1}=2^n\sum_{k=0}^{n}\binom{n+k+1}{2k+1}A_{3k-2n}, \tag{5}$$

where $A:=2^{j/2}\cos(j\pi/4)$ for all integers j. Using the addition theorem of the cosine, we easily find that, for all integers r,

$$A_{4r}=(-1)^r2^{2r},\;\; A_{4r+1}=(-1)^r2^{2r},\;\; A_{4r+2}=0,\;\; A_{4r+3}=(-1)^{r+1}2^{2r+1},$$

or, in a more compact form,

$$A_j=\begin{cases}(-1)^{[(j+1)/4]}2^{[j/2]}, & \text{if } j\not\equiv 2 \pmod 4,\\ 0, & \text{otherwise.}\end{cases} \tag{6}$$

Observing that $[(3k-2n)/2]=[3k/2]-n$, we see that (5) and (6) prove the stated equation with $n+1$ instead of n.

Reference

1. Ryshik & Gradstein. *Tafeln* $\sum\prod\int$ *Tables*. Berlin: VEB Deutscher Verlag der Wissenschaften, 1963.

Also solved by P. Bruckman

Editorial Note: The editor wishes to acknowledge that H.-J. Seiffert also solved H-504 and H-505.

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