JACOBSTHAL REPRESENTATION POLYNOMIALS

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1. PRELIMINARIES

Consider two sequences of polynomials $\{J_n(x)\}$, the *Jacobsthal polynomials*, and $\{j_n(x)\}$, the *Jacobsthal-Lucas polynomials*, defined recursively [3] by

$$J_{n+2}(x) = J_{n+1}(x) + 2xJ_n(x), \quad J_0(x) = 0, \ J_1(x) = 1, \tag{1.1}$$

and

$$j_{n+2}(x) = j_{n+1}(x) + 2xj_n(x), \quad j_0(x) = 2, \ j_1(x) = 1, \tag{1.2}$$

respectively.

Observe that $J_n(1/2) = F_n$ and $j_n(1/2) = L_n$, the n^{th} Fibonacci and Lucas numbers, respectively. When $x = 1$, we obtain the Jacobsthal and Jacobsthal-Lucas numbers [8], respectively. (Other number sequences derived from (1.1) and (1.2) which are of some interest are generated by $x = 1/4$.)

For $\{J_n(x)\}$ and $\{j_n(x)\}$, the characteristic equation is

$$\lambda^2 - \lambda - 2x = 0 \tag{1.3}$$

with roots

$$\left.\begin{aligned} \alpha(x) &= \frac{1+\sqrt{8x+1}}{2}, \\ \beta(x) &= \frac{1-\sqrt{8x+1}}{2}, \end{aligned}\right\} \tag{1.4}$$

so that

$$\left.\begin{aligned} \alpha(x)+\beta(x) &= 1, \\ \alpha(x)\beta(x) &= -2x, \\ \alpha(x)-\beta(x) &= \sqrt{8x+1} = \Delta(x), \end{aligned}\right\} \tag{1.5}$$

whence

$$\Delta(1) = 3. \tag{1.5a}$$

Moreover,

$$\left.\begin{aligned} \alpha^2(x)+2x &= \Delta(x)\alpha(x), \\ \beta^2(x)+2x &= -\Delta(x)\beta(x). \end{aligned}\right\} \tag{1.6}$$

Comparison might be made between our definition (1.1) and that in [2] for Jacobsthal polynomials. The correspondence is simple: x in [2] $\leftrightarrow$ $2x$ in (1.1). While the nomenclature in [2] serves a very valuable purpose leading to elegant results and extensions, we prefer to retain the factor $2x$ for consistency with our notation for Pell polynomials [10].

To the best of my knowledge, properties of the Jacobsthal-Lucas polynomials defined fully in (1.2), and the corresponding numbers [8] generated when $x = 1$, are generally due to the present author, as an appropriate companion to those of the Jacobsthal polynomials (1.1). (Our (3.10), (3.11), and (3.12) do occur in [12], though in a heavily camouflaged form.) When it is convenient (e.g., for brevity), the polynomials given by (1.1) and (1.2) will simply be referred to collectively as *Jacobsthal-type polynomials*, or, as in the title of the paper, more simply still as *Jacobsthal polynomials.*

Aspects of Jacobsthal polynomials (1.1), which are documented in other sources (e.g., [1], [2], [12]) will not in general be duplicated in this presentation, though the basic features must recur.

Goals of This Paper

Aims of this presentation are:

(i) to exhibit some basic properties of the polynomials (Tables 1 and 2) which generalize the properties of the corresponding numbers in [8];

(ii) to reveal some of the salient features of the diagonal functions generated by (1.1) and (1.2);

(iii) to examine the properties of the "augmented" polynomials developed from (1.1) and (1.2) by the addition of an appropriate constant.

2. THE JACOBSTHAL-TYPE POLYNOMIALS

Tables 1 and 2 list the first few polynomials of (1.1) and (1.2) of these Jacobsthal-type sequences.

TABLE 1. Jacobsthal Polynomials $\{J_n(x)\}: 0 \le n \le 10$

$J_0(x)=0$	$J_6(x)=1+8x+12x^2$
$J_1(x)=1$	$J_7(x)=1+10x+24x^2+8x^3$
$J_2(x)=1$	$J_8(x)=1+12x+40x^2+32x^3$
$J_3(x)=1+2x$	$J_9(x)=1+14x+60x^2+80x^3+16x^4$
$J_4(x)=1+4x$	$J_{10}(x)=1+16x+84x^2+160x^3+80x^4$
$J_5(x)=1+6x+4x^2$	

TABLE 2. Jacobsthal-Lucas Polynomials $\{j_n(x)\}: 0 \le n \le 10$

$j_0(x)=2$	$j_6(x)=1+12x+36x^2+16x^3$
$j_1(x)=1$	$j_7(x)=1+14x+56x^2+56x^3$
$j_2(x)=1+4x$	$j_8(x)=1+16x+80x^2+128x^3+32x^4$
$j_3(x)=1+6x$	$j_9(x)=1+18x+108x^2+240x^3+144x^4$
$j_4(x)=1+8x+8x^2$	$j_{10}(x)=1+20x+140x^2+400x^3+400x^4+64x^5$
$j_5(x)=1+10x+20x^2$	

Equivalent expressions for $\{J_n(x)\}$ in Table 1 are given in [2] with $x \leftrightarrow 2x$, as mentioned in Section 1.

3. BASIC PROPERTIES OF THE JACOBSTHAL-TYPE POLYNOMIALS

Generating Functions

$$\sum_{i=1}^{\infty} J_i(x)y^{i-1} = (1-y-2xy^2)^{-1}, \tag{3.1}$$

$$\sum_{i=1}^{\infty} j_i(x)y^{i-1} = (1+4xy)(1-y-2xy^2)^{-1}. \tag{3.2}$$

Binet Forms

$$J_n(x) = \frac{\alpha^n(x)-\beta^n(x)}{\Delta(x)}, \tag{3.3}$$

$$j_n(x) = \alpha^n(x)+\beta^n(x). \tag{3.4}$$

Simson Formulas

$$J_{n+1}(x)J_{n-1}(x) - J_n^2(x) = (-1)^n(2x)^{n-1}, \tag{3.5}$$

$$\left.\begin{aligned} j_{n+1}(x)j_{n-1}(x) - j_n^2(x) &= -\Delta^2(x)(-1)^n(2x)^{n-1} \\ &= -\Delta^2(x)(J_{n+1}(x)J_{n-1}(x) - J_n^2(x)) \end{aligned}\right\}. \tag{3.6}$$

Summation Formulas

$$\sum_{i=1}^{n} J_i(x) = \frac{J_{n+2}(x)-1}{2x}, \tag{3.7}$$

$$\sum_{i=0}^{n} j_i(x) = \frac{j_{n+2}(x)-1}{2x}. \tag{3.8}$$

Explicit Closed Forms

$$J_n(x) = \sum_{r=0}^{\left[\frac{n-1}{2}\right]} \binom{n-1-r}{r}(2x)^r, \tag{3.9}$$

$$j_n(x) = \sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{n}{n-r}\binom{n-r}{r}(2x)^r. \tag{3.10}$$

Important Interrelationships

$j_n(x)J_n(x) = J_{2n}(x)$	[by (3.3), (3.4)],	(3.11)
$j_n(x) = J_{n+1}(x) + 2xJ_{n-1}(x)$	[from (3.1), (3.2)],	(3.12)
$\Delta^2(x)J_n(x) = j_{n+1}(x) + 2xj_{n-1}(x)$	[by (1.6), (3.3), (3.4)],	(3.13)
$J_n(x) + j_n(x) = 2J_{n+1}(x)$	[a form of (3.12)],	(3.14)
$\Delta^2(x)J_n(x) + j_n(x) = 2j_{n+1}(x)$	[a form of (3.13)],	(3.15)
$\Delta(x)J_n(x) + j_n(x) = 2\alpha^n(x)$	[by (3.3), (3.4)],	(3.16)
$\Delta(x)J_n(x) - j_n(x) = -2\beta^n(x)$	[by (3.3), (3.4)],	(3.17)

$$J_m(x)j_n(x)+J_n(x)j_m(x)=2J_{m+n}(x) \qquad \text{[by (3.3), (3.4)]}, \tag{3.18}$$

$$j_m(x)j_n(x)+\Delta^2(x)J_m(x)J_n(x)=2j_{m+n}(x) \qquad \text{[by (3.3), (3.4)]}, \tag{3.19}$$

whence $(m=n)$

$$j_n^2(x)+\Delta^2(x)J_n^2(x)=2j_{2n}(x), \tag{3.20}$$

the left-hand side being a sum of squares. Putting $m=n$ in (3.18) reduces the formula to (3.11). Readers are invited to discover formulas corresponding to (3.18) and (3.19) when the + sign on the left-hand side is replaced by a – sign (leading in the second instance to a difference of squares).

A neat differentiation worth recording is

$$\frac{dj_n(x)}{dx}=2nJ_{n-1}(x), \tag{3.21}$$

which differs appreciably from analogous derivatives for other "Lucas-type" polynomials, namely, those for which the initial term (i.e., when $n=0$) has the value 2 (see [7]). Less exciting is the companion result

$$\Delta^2(x)\frac{dJ_n(x)}{dx}=2nj_{n-1}(x)-4J_n(x). \tag{3.22}$$

Column Generators of $\{J_n(x)\}$ and $\{j_n(x)\}$

Formulas (3.1) and (3.2) disclose the methods for producing the polynomials $\{J_n(x)\}$ and $\{j_n(x)\}$, i.e., the rows in Tables 1 and 2. Columns in Table 1 are readily seen to be generated by $(2xy^2)^0(1-y)^{-1}$, $(2xy^2)(1-y)^{-2}$, $(2xy^2)^2(1-y)^{-3}$, $(2xy^2)^3(1-y)^{-4}$, $(2xy^2)^4(1-y)^{-5}$, ..., i.e., the r^{th} column is born from

$$(2xy^2)^{r-1}(1-y)^{-r} \quad (r\ge 1). \tag{3.23}$$

The column generator for the r^{th} column in Table 2 is conceived to be

$$\begin{aligned}(2xy^2)^{r-1}&\left[\frac{1}{(1-y)^{r-1}}+\frac{1}{(1-y)^r}\right] \quad (r\ge 1)\\ &=(2xy^2)^{r-1}\frac{2-y}{(1-y)^r}.\end{aligned} \tag{3.24}$$

Associated Sequences

Suppose we define the k^{th} *associated sequences* $\{J_n^{(k)}(x)\}$ and $\{j_n^{(k)}(x)\}$ of $\{J_n(x)\}$ and $\{j_n(x)\}$ to be, respectively $(k\ge 1)$,

$$J_n^{(k)}(x)=J_{n+1}^{(k-1)}(x)+2xJ_{n-1}^{(k-1)}(x) \tag{3.25}$$

and

$$j_n^{(k)}(x)=j_{n+1}^{(k-1)}(x)+2xj_{n-1}^{(k-1)}(x), \tag{3.26}$$

where $J_n^{(0)}(x)=J_n(x)$ and $j_n^{(0)}(x)=j_n(x)$. Accordingly,

$$J_n^{(1)}(x)=j_n(x) \quad \text{[by (3.12)]} \tag{3.27}$$

and

$$j_n^{(1)}(x) = \Delta^2(x) J_n(x) \quad \text{[by (3.13)]} \tag{3.28}$$

are the generic members of the *first associated sequences* $\{J_n^{(1)}(x)\}$ and $\{j_n^{(1)}(x)\}$.

Repeated manipulation of the above formulas eventually reveals that

$$\begin{cases} J_n^{(2m)}(x) = j_n^{(2m-1)}(x) = \Delta^{2m}(x) J_n(x), \\ J_n^{(2m+1)}(x) = j_n^{(2m)}(x) = \Delta^{2m}(x) j_n(x). \end{cases} \tag{3.29}$$

Thus, for $m = 1, n = 5$,

$$\begin{cases} J_5^{(2)}(x) = j_5^{(1)}(x) = (8x+1)J_5(x) = 1 + 14x + 52x^2 + 32x^3, \\ J_5^{(3)}(x) = j_5^{(2)}(x) = (8x+1)j_5(x) = 1 + 18x + 100x^2 + 160x^3. \end{cases}$$

Another approach [7] may be employed to discover the formulas (3.29).

4. DIAGONAL FUNCTIONS

Inherent in the structure of $\{J_n(x)\}$ and $\{j_n(x)\}$ are the rising and descending diagonals which are fashioned in a manner analogous to those for Chebyshev and Fermat polynomials [4], [5].

Rising Diagonals

Imagine parallel upward-slanting lines in Tables 1 and 2 in which there exist the *rising diagonal functions* $\{R_1(x)\}$ and $\{r_1(x)\}$, respectively. Some of these are, say,

$$R_0(x) = 0,\ R_1(x) = R_2(x) = R_3(x) = 1,\ R_4(x) = 1 + 2x, \ldots, R_{10}(x) = 1 + 14x + 40x^2 + 8x^3 \tag{4.1}$$

and

$$r_0(x) = 2,\ r_1(x) = r_2(x) = 1,\ r_3(x) = 1 + 4x, \ldots, r_{10}(x) = 1 + 18x + 80x^2 + 56x^3. \tag{4.2}$$

Generating functions unfold by the usual technique. We have

$$\sum_{i=1}^{\infty} R_i(x) t^{i-1} = (1 - t - 2xt^3)^{-1} \tag{4.3}$$

and

$$\sum_{i=0}^{\infty} r_i(x) t^i = (2 - t)(1 - t - 2xt^3)^{-1}. \tag{4.4}$$

Alternatively, see (4.10),

$$\sum_{i=1}^{\infty} r_i(x) t^{i-1} = (1 + 4xt^2)(1 - t - 2xt^3)^{-1}. \tag{4.4$'$}$$

Comparing (4.3) with (4.4), and taking into account the different initial values of i therein, we arrive at

$$r_n(x) = 2R_{n+1}(x) - R_n(x), \tag{4.5}$$

i.e.,

$$r_n(x) + R_n(x) = 2R_{n+1}(x), \tag{4.5$'$}$$

which bears a formal correspondence with (3.14).

Inherent in (4.3) and (4.4) are the recurrence relations ($n \geq 3$)

$$R_n(x) = R_{n-1}(x) + 2xR_{n-3}(x) \tag{4.6}$$

and

$$r_n(x) = r_{n-1}(x) + 2xr_{n-3}(x). \tag{4.7}$$

Explicit closed forms are

$$R_n(x) = \sum_{r=0}^{\left[\frac{n-1}{3}\right]} \binom{n-1-2r}{r} (2x)^r \tag{4.8}$$

and

$$r_n(x) = 1 + \sum_{r=1}^{\left[\frac{n}{3}\right]} \frac{n-r}{r} \binom{n-1-2r}{r-1} (2x)^r. \tag{4.9}$$

Recall from (1.5) that $2x = -\alpha(x)\beta(x)$, so that (4.8) and (4.9) allow us to express $R_n(x)$ and $r_n(x)$ in terms of $\alpha(x)$ and $\beta(x)$.

Combinatorial calculations (including Pascal's formula) may be employed to establish (4.8) and (4.9) from the recurrence formulas. Proofs by inductive methods may also be applied, but these are somewhat tortuous and are omitted as a leisure activity for the dedicated reader who can convert a tedious activity into a pleasurable challenge.

From (4.5) and (4.6), it follows immediately ($n \to n+1$) that

$$r_n(x) = R_n(x) + 4xR_{n-2}(x) \quad (n \geq 2). \tag{4.10}$$

This result also follows directly from (4.4)$'$. Combining (4.5)$'$ and (4.10), we deduce that

$$r_n^2(x) - R_n^2(x) = 8xR_{n+1}(x)R_{n-2}(x). \tag{4.11}$$

Oddly, there is no result like (4.10) in which $R_n(x)$ (possibly with a factor) and $r_n(x)$ are interchanged, as in (4.6) and (4.7), for descending diagonals. (Why is this so?) A similar situation exists for Pell-type polynomials (cf. [13]).

Differential equations (partial) of the first order are readily determined from (4.3) and (4.4) on writing

$$R \equiv R(x, t) = \sum_{i=1}^{\infty} R_i(x) t^{i-1} \tag{4.12}$$

and

$$r \equiv r(x, t) = \sum_{i=0}^{\infty} r_i(x) t^i. \tag{4.13}$$

These are

$$2t^3 \frac{\partial R}{\partial t} - (1 + 6xt^2) \frac{\partial R}{\partial x} = 0 \tag{4.14}$$

and

$$2t^3 \left(\frac{\partial r}{\partial t} + R \right) - (1 + 6xt^2) \frac{\partial r}{\partial x} = 0. \tag{4.15}$$

Theoretically, there exists a pair of ordinary differential equations derivable from $R_n(x)$ and $r_n(x)$ (see [4], [5]), but so far their nature has not been vouchsafed to the writer.

Coming now to descending diagonal polynomials, we encounter a surprisingly felicitous ease with the mathematics (as also occurs, e.g., in [4], [5]).

Descending Diagonals

Formed in a similar way to the rising diagonal polynomials, except that we now imagine systems of parallel downward-slanting lines (cf. [4] for Chebyshev and Fermat polynomials), we behold the *descending diagonal functions* $\{D_i(x)\}$ and $\{d_i(x)\}$, respectively.

Some of these are, say,

$$\begin{aligned} &D_0(x)=0,\ D_1(x)=1,\ D_2(x)=1+2x,\ \dots,\\ &D_5(x)=1+8x+24x^2+32x^3+16x^4 \end{aligned} \tag{4.16}$$

and

$$\begin{aligned} &d_0(x)=2,\ d_1(x)=1+4x,\ d_2(x)=1+6x+8x^2,\ \dots,\\ &d_5(x)=1+12x+56x^2+128x^3+144x^4+64x^5. \end{aligned} \tag{4.17}$$

Patterns of behavior are readily discernible from the formation of the generating functions

$$\sum_{n=1}^{\infty} D_n(x)t^{n-1}=[1-(1+2x)t]^{-1}, \tag{4.18}$$

and

$$\sum_{n=1}^{\infty} d_n(x)t^{n-1}=(1+4x)[1-(1+2x)t]^{-1}, \tag{4.19}$$

whence $(n\geq 1)$

$$D_n(x)=(1+2x)^{n-1} \tag{4.20}$$

and

$$d_n(x)=(1+4x)(1+2x)^{n-1}, \tag{4.21}$$

leading to

$$d_n(x)=(1+4x)D_n(x), \tag{4.22}$$

$$\frac{D_n(x)}{D_{n-1}(x)}=\frac{d_n(x)}{d_{n-1}(x)}=1+2x \quad (n\geq 2), \tag{4.23}$$

i.e., $d_n(x)D_{n-1}(x)=d_{n-1}(x)D_n(x)$,

$$\frac{d_n(x)}{D_n(x)}=1+4x \quad (n\geq 1), \tag{4.24}$$

$$d_n(x)=D_{n+1}(x)+2xD_n(x), \tag{4.25}$$

$$(1+4x)^2D_n(x)=d_{n+1}(x)+2xd_n(x), \tag{4.26}$$

$$J_5(x)D_{n-1}(x)=D_{n+1}(x)+2xD_{n-1}(x), \tag{4.27}$$

and

$$J_5(x)d_{n-1}(x)=d_{n+1}(x)+2xd_{n-1}(x). \tag{4.28}$$

See Table 1 for $J_5(x)$. *En passant*, notice that $1+2x$ and $1+4x$, occurring in (4.18)-(4.24) and (4.26), may be expressed variously in terms of polynomials in Table 1, Table 2, (4.1), (4.2), (4.16), and (4.17).

Observe that the summed forms in (4.27)-(4.28) preclude the possibility of any associated sequence properties of $\{D_n(x)\}$ and $\{d_n(x)\}$ analogous to those for $\{J_n(x)\}$ and $\{j_n(x)\}$. (Put $k=1$ in (3.25) and (3.26) for the comparison and contrast.)

Quartet: Differential Equations

Write

$$D \equiv D(x,t) = \sum_{n=1}^{\infty} D_n(x)t^{n-1} = [1-(1+2x)t]^{-1} \tag{4.29}$$

and

$$d \equiv d(x,t) = \sum_{n=1}^{\infty} d_n(x)t^{n-1} = (1+4x)[1-(1+2x)t]^{-1} \tag{4.30}$$

using (4.5) and (4.6).

Without difficulty one derives, from (4.20), (4.21), (4.29), and (4.30),

$$2t\frac{\partial D}{\partial t} - (1+2x)\frac{\partial D}{\partial x} = 0, \tag{4.31}$$

$$2t\frac{\partial d}{\partial t} - (1+2x)\left\{\frac{\partial d}{\partial x} - 4D\right\} = 0, \tag{4.32}$$

$$(1+2x)\frac{dD_n}{dx}(x) = 2(n-1)D_n(x) \tag{4.33}$$

$$\frac{dd_n(x)}{dx} = 2\{2D_n(x) + (n-1)d_{n-1}(x)\}. \tag{4.34}$$

More generally,

$$(1+2x)^k \frac{\partial^k D}{\partial x^k} = (2t)^k \frac{\partial^k D}{\partial t^k} = \frac{k!\{(1+2x)2t\}^k}{[1-(1+2x)t]^{k+1}} \tag{4.35}$$

and

$$\frac{d^{n-1}D_n(x)}{dx^{n-1}} = (n-1)!2^{n-1}. \tag{4.36}$$

Roots

Clearly, from (4.20) and (4.21), the polynomial equations $D_n(x)=0$ (of degree $n-1$) and $d_n(x)=0$ (of degree n) have multiple roots, namely, an $(n-1)$-fold root $-1/2$ in the first case and an $(n-1)$-fold root $-1/2$ together with a root $x=-1/4$ in the second case.

Diagonal Numbers

Substitute $x=1$ in (4.1), (4.2), (4.16), and (4.17). Then the skeletal profiles of the bodies fleshed out by the polynomials reduce to

n	0	1	2	3	4	5	6	7	8	9	10	11
$\{R_n\}$	0	1	1	1	3	5	7	13	23	37	63	109
$\{r_n\}$	2	1	1	5	7	9	19	33	51	89	155	267
$\{D_n\}$	0	1	3	9	27	81	243	729	2187	⋯	⋯	⋯
$\{d_n\}$	2	5	15	45	135	405	1215	3645	⋯	⋯	⋯	⋯

(4.37)

So, e.g., by (4.6), (4.7), (4.10), (4.22), and (4.23),

$$\left.\begin{array}{l} R_n = R_{n-1} + 2R_{n-3}, \\ r_n = r_{n-1} + 2r_{n-3}, \\ r_n = R_n + 4R_{n-2}, \\ d_n = 5D_n = 15D_{n-1} \ \text{since}\ D_n = 3D_{n-1}. \end{array}\right\} \tag{4.38}$$

Diagonal numbers for (say) Fibonacci, Pell, Fermat, and Chebyshev polynomials *inter alia* could be tabulated, along with the numbers for their cognate "Lucas" polynomials. See, e.g., [4], [5], [11], [13], and [14].

Reverting to (4.1), (4.2), (4.16), and (4.17), we may find mild interest in substituting $x = 1/2$ and $x = 1/4$.

Bizarre Afterthought

What of any interest might eventuate if we imagined rising rising diagonals, descending descending diagonals, rising descending diagonals, and other combinations of the two elementary dichotomous concepts of rising and (falling) descending?

Conjectures

$$R_{m+n}(x) = R_{m+1}(x)R_n(x) + 2xR_m(x)R_{n-2}(x) + 2xR_{m-1}(x)R_{n-1}(x) \tag{4.39}$$

and

$$r_{m+n}(x) = R_{m+1}(x)r_n(x) + 2xR_m(x)r_{n-2}(x) + 2xR_{m-1}(x)r_{n-1}(x). \tag{4.40}$$

5. AUGMENTED JACOBSTHAL-TYPE REPRESENTATION POLYNOMIALS

New symbolism and terminology are now required.

Following the situation for the number sequences $\{\mathcal{T}_n\}$ and $\{\hat{j}\}$ described in (3.4) and (3.5) of [8], we introduce the *augmented Jacobsthal representation polynomial sequence* $\{J_n(x)\}$ defined by

$$\mathcal{T}_{n+2}(x) = \mathcal{T}_{n+1}(x) + 2x\mathcal{T}_n(x) + 3, \quad \mathcal{T}_0(x) = 0, \ \mathcal{T}_1(x) = 1, \tag{5.1}$$

and the *augmented Jacobsthal-Lucas representation polynomial sequence* $\{\hat{j}_n(x)\}$ defined by

$$\hat{j}_{n+2}(x) = \hat{j}_{n+1}(x) + 2x\hat{j}_n(x) + 5, \quad \hat{j}_0(x) = 0, \ \hat{j}_1(x) = 1. \tag{5.2}$$

Some of these are, for example,

$$\mathcal{T}_0(x) = 0,\ \mathcal{T}_1(x) = 1,\ \mathcal{T}_2(x) = 4,\ \mathcal{T}_3(x) = 7 + 2x, \ldots,\ \mathcal{T}_8(x) = 22 + 102x + 160x^2 + 56x^3 \tag{5.3}$$

and

$$\hat{j}_0(x) = 0,\ \hat{j}_1(x) = 1,\ \hat{j}_2(x) = 6,\ \hat{j}_3(x) = 11 + 2x, \ldots,\ \hat{j}_8(x) = 36 + 162x + 240x^2 + 72x^3. \tag{5.4}$$

The choice and the *raison d'être* of the constants +3 and +5 in (5.1) and (5.2) are explained in [8]. Properties of these new polynomial sequences $\{\mathcal{J}_n(x)\}$ and $\{\hat{j}_n(x)\}$ are worthy of consideration *per se*.

Replacing +3 and +5 more generally by $+c$ has been done in a separate paper which thus covers the four special polynomial sequences $\{J_n(x)\}$, $\{j_n(x)\}$, $\{\mathcal{J}_n(x)\}$, and $\{\hat{j}_n(x)\}$.

6. BASIC PROPERTIES OF $\{\mathcal{J}_n(x)\}$ and $\{\hat{j}_n(x)\}$

Generating Functions

Standard techniques lead readily to

$$\sum_{i=1}^{\infty} \mathcal{J}_i(x)y^{i-1} = \frac{1+2y}{1-2y-(2x-1)y^2+2xy^3}, \tag{6.1}$$

$$\sum_{i=1}^{\infty} \hat{j}_i(x)y^{i-1} = \frac{1+4y}{1-2y-(2x-1)y^2+2xy^3}. \tag{6.2}$$

Binet Forms

Examination of Table 1 and (5.3) leads to the somewhat surprising observation that

$$\mathcal{J}_n(x) = \frac{J_{n+2}(x)+2J_{n+1}(x)-3}{2x}. \tag{6.3}$$

Proof of (6.3): Checking quickly validates the cases $n = 0, 1, 2, 3, 4$ (say). Assume (6.3) is true for $n = k$ (fixed integer), i.e., suppose

$$\mathcal{J}_k(x) = \frac{J_{k+2}(x)+2J_{k+1}(x)-3}{2x} \quad \cdots \ (H).$$

Then

$$\begin{aligned}\mathcal{J}_{k+1}(x) &= \mathcal{J}_k(x)+2x\mathcal{J}_{k-1}(x)+3 \quad \text{by (5.1)} \\ &= \frac{J_{k+2}(x)+2J_{k+1}(x)-3+2x[J_{k+1}(x)+2J_k(x)-3]}{2x}+3 \\ &= \frac{J_{k+2}(x)+2xJ_{k+1}(x)+2\{J_{k+1}(x)+2xJ_k(x)\}-3}{2x} \\ &= \frac{J_{k+3}(x)+2J_{k+2}(x)-3}{2x},\end{aligned}$$

where the hypothesis (H) has been applied.

Hence, (6.3) is true for $n = k+1$, and so, for all n.

Consequently, (6.3) is true, by induction.

Induction is used in a similar fashion to establish

$$\hat{j}_n(x) = \frac{J_{n+2}(x)+4J_{n+1}(x)-5}{2x}. \tag{6.4}$$

For example,

$$
\begin{aligned}
n=7 \Rightarrow \text{R.H.S.} &= \frac{(1+14x+60x^2+80x^3+16x^4)+4(1+12x+40x^2+32x^3)-5}{2x} \\
&= 31+110x+104x^2+8x^3 \\
&= \hat{j}_7(x) \quad \text{from Table 1 and (5.4).}
\end{aligned}
$$

Binet forms for $\mathcal{T}_n(x)$ and $\hat{j}_n(x)$ are obtainable by substituting for $J_n(x)$ from (3.3) in (6.3) and (6.4).

Simson Formulas

$$\mathcal{T}_{n+1}(x)\,\mathcal{T}_{n-1}(x)-\mathcal{T}_n^2(x)=(-2x)^{n-2}(2x-6)-3(J_{n-1}(x)+2J_{n-2}(x)), \tag{6.5}$$

$$\hat{j}_{n+1}(x)\hat{j}_{n-1}(x)-\hat{j}_n^2(x)=(-2x)^{n-2}(2x-20)-5(J_{n-1}(x)+4J_{n-2}(x)). \tag{6.6}$$

Summation Formulas

$$\sum_{i=1}^{n}\mathcal{T}_i(x)=\frac{\mathcal{T}_{n+2}(x)-3n-4}{2x}, \tag{6.7}$$

$$\sum_{i=1}^{n}\hat{j}_i(x)=\frac{\hat{j}_{n+2}(x)-5n-6}{2x}. \tag{6.8}$$

Explicit Closed Forms

$$\mathcal{T}_n(x) = J_n(x)+3\sum_{r=0}^{\left[\frac{n-1}{2}\right]}\binom{n-1-r}{r+1}(2x)^r, \tag{6.9}$$

$$\hat{j}_n(x)=J_n(x)+5\sum_{r=0}^{\left[\frac{n-1}{2}\right]}\binom{n-1-r}{r+1}(2x)^r. \tag{6.10}$$

Spotting the second portion of the expressions in (6.9) was not easy. Induction provides us with a proof.

Proof of (6.9): Verification of (6.9) for $n=1,2,3$ is straightforward. Assume (6.9) is true for $n=1,2,3,\dots,k-1,k$. Then, by (1.1) and the hypothesis,

$$
\begin{aligned}
\mathcal{T}_k(x)+2x\mathcal{T}_{k-1}(2)+3 &= J_{k+1}(x)+3\left\{1+\sum_{r=0}^{\left[\frac{k-1}{2}\right]}\binom{k-1-r}{r+1}(2x)^r+\sum_{r=0}^{\left[\frac{k-2}{2}\right]}\binom{k-2-r}{r+1}(2x)^{r+1}\right\} \\
&= J_{k+1}(x)+3\sum_{r=0}^{\left[\frac{k}{2}\right]}\binom{k-r}{r+1}(2x)^r \\
&= \mathcal{T}_{k+1}(x) \quad \text{by (6.9).}
\end{aligned}
$$

Being valid for $n=k+1$, the theorem is true for all n.

Pascal's formula for binomial coefficients has been applied in the proof of (6.9) when combining corresponding powers $(2x)^r$, $r=1,2,3,\dots$. Also, we have absorbed $1=\binom{k-1}{0}$ into the constant $\binom{k-1}{1}$ to produce $\binom{k}{1}$, by Pascal's formula.

Arguments of a similar nature are applicable for (6.10).

Observe the simple, but important, connection between (6.3) and (6.4):

$$\hat{j}_n(x) - \mathcal{T}_n(x) = \frac{J_{n+1}(x) - 1}{x}. \tag{6.11}$$

This seems to be an appropriate place at which to conclude our theory, though more could be told.

7. CONCLUDING REMARKS

Possibilities for other avenues of development that present themselves include, for example:

(i) the extension of the theory in this paper to negative subscripts [9];

(ii) convolutions for Jacobsthal-type polynomials (cf. [13]);

(iii) further work on diagonal functions, e.g., as in [14];

(iv) research into Jacobsthal-type polynomials along the lines of that for Pell-type polynomials in [13] and in a series of papers by Mahon and Horadam, e.g., [10].

Initial exploration of some of these opportunities has commenced.

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