ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Stanley Rabinowitz

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent by electronic mail to Fibonacci@MathPro.com *on Internet. All correspondence will be acknowledged.*

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions. Proposers should inform us of the history of the problem if it is not original. A problem should not be submitted elsewhere while it is under consideration for publication in this column.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-836 ***Proposed by Al Dorp, Edgemere, NY***

Replace each of "W", "X", "Y", and "Z" by either "F" or "L" to make the following an identity:

$$W_n^2 - 6X_{n+1}^2 + 2Y_{n+2}^2 - 3Z_{n+3}^2 = 0.$$

B-837 ***Proposed by Joseph J. Koštál, Chicago, IL***

Let

$$P(x) = x^{1997} + x^{1996} + x^{1995} + \cdots + x^2 + x + 1$$

and let $R(x)$ be the remainder when $P(x)$ is divided by $x^2 - x - 1$. Show that $R(x)$ is divisible by F_{999}.

B-838 ***Proposed by Peter G. Anderson, Rochester Institute of Technology, Rochester, NY***

Define a sequence of linear polynomials, $f_n(x) = m_n x + b_n$, by the recurrence

$$f_n(x) = f_{n-1}(f_{n-2}(x)),\quad n \geq 3,$$

with initial conditions

$$f_1(x) = \frac{1}{2}x$$

and

$$f_2(x) = \frac{1}{2}x + \frac{1}{2}.$$

Find a formula for m_n.

Extra credit: Find a formula for b_n.

B-839 ***Proposed by David M. Bloom, Brooklyn College of CUNY, Brooklyn, NY***

Evaluate the sum

$$\sum_{k=0}^{\lfloor n/3 \rfloor} (-1)^k 2^{-3k} \binom{n-2k}{k}$$

in terms of Fibonacci numbers.

B-840 ***Proposed by the editor***

Let

$$A_n = \begin{pmatrix} F_n & L_n \\ L_n & F_n \end{pmatrix}.$$

Find a formula for A_{2n} in terms of A_n and A_{n+1}.

B-841 ***Proposed by David Zeitlin, Minneapolis, MN***

Let P be an integer. For $n \ge 0$, let $U_{n+2} = PU_{n+1} + U_n$, with $U_0 = 0$ and $U_1 = 1$. Also let $V_{n+2} = PV_{n+1} + V_n$, with $V_0 = 2$ and $V_1 = P$. Prove that

$$\frac{V_n^2 + V_{n+a}^2}{U_n^2 + U_{n+a}^2}$$

is always an integer if a is odd.

NOTE: The Elementary Problems Column is in need of more **easy**, yet elegant and non-routine problems.

SOLUTIONS

Nonstandard Recurrence

B-820 ***Proposed by the editor; Dedicated to Herta Freitag (Vol. 34, no. 5, November 1996)***

Find a recurrence (other than the usual one) that generates the Fibonacci sequence.

[The usual recurrence is a second-order linear recurrence with constant coefficients. Can you find a first-order recurrence that generates the Fibonacci sequence? Can you find a third-order linear recurrence? a nonlinear recurrence? one with nonconstant coefficients? etc.]

There were so many fine formulas sent in that we will only list a selection of them. We omit the obvious initial conditions.

First-Order Recurrences

$F_{n+1} = \alpha F_n + \beta^n$. — Dresel

$F_{n+1} = \beta F_n + \alpha^n$. — Bruckman

$F_{n+1} = \lfloor \alpha F_n + 0.4 \rfloor, \; n \geq 2.$ Dresel

$F_{n+1} = \frac{1}{2}(F_n + \sqrt{5F_n^2 + 4(-1)^n}).$ Dresel

Second-Order Nonlinear Recurrences

$F_{n+1} = ((-1)^n + F_n^2) / F_{n-1}.$ Anderson/Bruckman

Third-Order Recurrences

$F_{n+3} = 2F_{n+1} + F_n.$ Hendel

$F_{n+3} = 2F_{n+2} - F_n.$ Bruckman

$F_{n+3} = [pF_{n+2} + (p+2q)F_{n+1} + qF_n] / (p+q).$ Dresel

Fourth-Order Recurrences

$F_{n+4} = (1+p+q)F_{n+3} + (1-p-q-pq)F_{n+2} + (pq-p-q)F_{n+1} + pqF_n.$ Taylor

k^{th}-Order Linear Recurrences with Constant Coefficients

$F_n = F_{n-k} + \sum_{j=1}^{k} F_{n-j-1}.$ Anderson

$F_n = F_{n-2k+2} + \sum_{j=1}^{k-1} F_{n+2j-2k+1}.$ Freitag

Other Nonlinear Recurrences

$F_n = \sqrt[3]{F_{n-1}^3 + 3F_{n-1}^2 F_{n-2} + F_{n-2}^3 + 4F_{n-1}F_{n-2}^2}.$ Anderson/Seiffert

$F_n = \frac{1}{2}\left(L_k F_{n-k} + F_k \sqrt{5F_{n-k}^2 + 4(-1)^{n-k}}\right).$ Seiffert

$F_n = (F_{n-k}^2 - (-1)^n F_k^2) / F_{n-2k}.$ Seiffert

$F_n = F_{\lfloor (n+2)/2 \rfloor}^2 - (-1)^n F_{\lfloor (n-1)/2 \rfloor}^2.$ Seiffert

Taylor sent an 11-page tome of formulas, too many to list them all.

Also solved by Peter G. Anderson, Paul S. Bruckman, Leonard A. G. Dresel, Herta T. Freitag, Russell Jay Hendel, H.-J. Seiffert, and Joan Marie Taylor.

Fibonacci Rectangle

B-821 ***Proposed by L. A. G. Dresel, Reading, England (Vol. 35, no. 1, February 1997)***

Consider the rectangle with sides of lengths F_{n-1} and F_{n+1}. Let A_n be its area, and let d_n be the length of its diagonal. Prove that $d_n^2 = 3A_n \pm 1$.

Solution by Steve Scarborough, Loyola Marymount University, Los Angeles, CA

$$\begin{aligned} d_n^2 - 3A_n &= F_{n-1}^2 + F_{n+1}^2 - 3F_{n-1}F_{n+1} \\ &= F_{n-1}^2 + (F_n + F_{n-1})^2 - 3F_{n-1}(F_n + F_{n-1}) \\ &= F_n^2 - F_{n-1}^2 - F_{n-1}F_n = F_n^2 - F_{n-1}^2 - F_{n-1}(F_{n+1} - F_{n-1}) \\ &= F_n^2 - F_{n-1}F_{n+1} = (-1)^{n+1}. \end{aligned}$$

The last step is Hoggatt's identity (I_{13}) from [1].

Reference

1. Verner E. Hoggatt, Jr.. *Fibonacci and Lucas Numbers.* Santa Clara, Calif.: The Fibonacci Association, 1979.

Also solved by Peter G. Anderson, Michel A. Ballieu, Brian D. Beasley, Scott H. Brown, Paul S. Bruckman, Charles K. Cook, Steve Edwards, Russell Euler & Jawad Sadek, Herta T. Freitag, Hans Kappus, Daina A. Krigens, Harris Kwong, Carl Libis, Bob Prielipp, Don Redmond, Maitland A. Rose, H.-J. Seiffert, Sahib Singh, Lawrence Somer, I. Strazdins, and the proposer.

A Tricky n^{th} Root

B-822 ***Proposed by Anthony Sofo, Victoria University of Technology, Australia (Vol. 35, no. 1, February 1997)***

For $n > 0$, simplify

$$\sqrt[n]{\alpha F_n + F_{n-1}} + (-1)^{n+1} \sqrt[n]{F_{n-1} - \alpha F_n}\,.$$

Solution by Hans Kappus, Rodersdorf, Switzerland

It is well known (see, for example, page 34 of [1]) that $\alpha F_n + F_{n-1} = \alpha^n$ and $F_{n+1} - \alpha F_n - \beta^n$. Ignoring complex n^{th} roots, we presume that the symbol $\sqrt[n]{x}$ denotes the principal root of the real quantity x. Since $\alpha > 0$ and $\beta < 0$, we have

$$\sqrt[n]{\alpha F_n + F_{n-1}} = \alpha$$

and

$$\sqrt[n]{F_{n+1} - \alpha F_n} = \sqrt[n]{\beta^n} = (-1)^n |\beta| = (-1)^{n+1} \beta.$$

Therefore, the expression given in the problem has value $\alpha + \beta = 1$.

Reference

1. Verner E. Hoggatt, Jr. *Fibonacci and Lucas Numbers.* Santa Clara, Calif: The Fibonacci Association, 1979.

Several readers incorrectly assumed that $\sqrt[n]{\beta^n} = \beta$, which is not true when n is even. Haukkanen found several analogs, such as $\sqrt[n]{F_{n+1} - \beta F_n} + (-1)^{n+1} \sqrt[n]{\beta F_n + F_{n-1}} = 1$.

Also solved by Brian D. Beasley, Paul S. Bruckman, Leonard A. G. Dresel, Herta T. Freitag, Pentti Haukkanen, Bob Prielipp, Steve Scarborough, H.-J. Seiffert, Lawrence Somer, and the proposer.

Solving a Simple Recurrence

B-823 ***Proposed by Pentti Haukkanen, University of Tampere, Finland (Vol. 35, no. 1, February 1997)***

It is easy to see that the solution of the recurrence relation

$$A_{n+2} = -A_{n+1} + A_n,\ A_0 = 0,\ A_1 = 1,$$

can be written as $A_n = (-1)^{n+1} F_n$.

Find a solution to the recurrence

$$A_{n+2} = -A_{n+1} + A_n,\ A_0 = 1,\ A_1 = 1,$$

in terms of F_n and L_n.

Solution by Hans Kappus, Rodersdorf, Switzerland

Let $A_n = (-1)^n B_n$. Then, for the B_n, we have the recurrence

$$B_{n+2} = B_{n+1} + B_n,\ B_0 = 1,\ B_1 = -1.$$

Hence, $B_n = aF_n + bL_n$ with constants a and b determined by the initial conditions, i.e., $2b = 1$ and $a + b = -1$. The result is

$$A_n = (-1)^n (L_n - 3F_n)/2.$$

An equivalent form of the answer is $A_n = (-1)^{n+1} F_{n-2}$.

Also solved by Michel A. Ballieu, Brian D. Beasley, Paul S. Bruckman, Charles K. Cook, Aloysius Dorp, Leonard A. G. Dresel, Russell Euler & Jawad Sadek, Daina A. Krigens, Harris Kwong, Carl Libis, Bob Prielipp, Maitland A. Rose, H.-J. Seiffert, Sahib Singh, Lawrence Somer, I. Strazdins, and the proposer.

Solving a Harder Recurrence

B-824 ***Proposed by Brian D. Beasley, Presbyterian College, Clinton, SC (Vol. 35, no. 1, February 1997)***

Fix a nonnegative integer m. Solve the recurrence $A_{n+2} = L_{2m+1}A_{n+1} + A_n$, for $n \geq 0$, with initial conditions $A_0 = 1$ and $A_1 = L_{2m+1}$, expressing your answer in terms of Fibonacci and/or Lucas numbers.

Solution by Paul S. Bruckman, Highwood, IL

The characteristic polynomial of the given recurrence is given by

$$p(z) = z^2 - L_{2m+1}z - 1 = (z - \alpha^{2m+1})(z - \beta^{2m+1}).$$

Therefore, there exist constants A and B, dependent solely on the initial conditions, such that

$$A_n = A\alpha^{(2m+1)n} + B\beta^{(2m+1)n}.$$

Setting $n = 0$ and $n = 1$, we obtain $A + B = 1$ and $A\alpha^{2m+1} + B\beta^{2m+1} = L_{2m+1}$. Solving this pair of equations, we obtain $A = D\alpha^{2m-1}$ and $B = -D\beta^{2m+1}$, where $D = (\alpha^{2m+1} - \beta^{2m+1})^{-1}$.

Thus,

$$A_n = D[\alpha^{(2m+1)(n+1)} - \beta^{(2m+1)(n+1)}],$$

which simplifies to

$$A_n = F_{(2m+1)(n+1)} / F_{2m+1}.$$

Several readers pointed out that the result follows from Problem B-748.

Also solved by Charles K. Cook, Leonard A. G. Dresel, Steve Edwards, Russell Euler & Jawad Sadek, Herta T. Freitag, Hans Kappus, Daina A. Krigens, Harris Kwong, Carl Libis, Don Redmond, H.-J. Seiffert, Lawrence Somer, and the proposer.

Divisors of Lucas Sequences

B-825 ***Proposed by Lawrence Somer, University of America, Washington, D.C. (Vol. 35, no. 1, February 1997)***

Let $\langle V_n \rangle$ be a sequence defined by the recurrence $V_{n+2} = PV_{n+1} - QV_n$, where P and Q are integers and $V_0 = 2$, $V_1 = P$. The integer d is said to be a divisor of $\langle V_n \rangle$ if $d | V_n$ for some $n \geq 1$.

(a) If P and Q are both even, show that 2^m is a divisor of $\langle V_n \rangle$ for any $m \geq 1$.

(b) If P or Q is odd, show that there exists a fixed nonnegative integer k such that 2^k is a divisor of $\langle V_n \rangle$ but 2^{k+1} is not a divisor of $\langle V_n \rangle$. If exactly one of P or Q is even, show that $2^k | V_1$; if P and Q are both odd, show that $2^k | V_3$.

Solution by the proposer

First, suppose that P and Q are both even. Using the recursion relation defining $\langle V_n \rangle$, it follows by induction that $2^{i+1} | V_{2i}$ and $2^{i+1} | V_{2i+1}$ for $i \geq 0$. Thus, 2^m is a divisor of $\langle V_n \rangle$ for all $m \geq 1$.

Now, suppose that $2 | Q$ but $2 \nmid P$. One sees by induction that V_n is odd for all $n \geq 1$. Hence, $2^0 = 1$ is a divisor of $\langle V_n \rangle$ but $2^1 = 2$ is not a divisor of $\langle V_n \rangle$. Clearly, $2^0 | V_1$.

We now assume that $2^k \| P$ but $2 \nmid Q$, where $k \geq 1$. It follows by induction that $2^k \| V_{2n-1}$ and $2 \| V_{2n}$ for $n \geq 1$. Then 2^k is a divisor of $\langle V_n \rangle$ but 2^{k+1} is not a divisor of $\langle V_n \rangle$. Clearly, $2^k | V_1$.

Finally, assume that P and Q are both odd. By inspection, one sees that $2 | V_n$ if and only if $3 | n$. Suppose that $2^k \| V_3$, where $k \geq 1$. By the Binet formula, $V_n = \gamma^n + \delta^n$, where γ and δ are roots of the equation $x^2 - Px + Q = 0$. Consider the sequence $\langle V_n' \rangle$ defined by $V_n' = V_{3n}$. Then

$$V_n' = \gamma^{3n} + \delta^{3n} = (\gamma^3)^n + (\delta^3)^n,$$

where γ^3 and δ^3 are roots of the equation $x^2 - V_3 x + Q^3 = 0$. Thus, $\langle V_n' \rangle$ is a Lucas sequence of the second kind satisfying the second-order linear recursion relation

$$V_{n+2}' = P' V_{n+1}' - Q' V_n',$$

where $P' = V_3$ is even, $Q' = Q^3$ is odd, $V_0' = 2$, and $V_1' = P' = V_3$. Hence, $2^k \| V_1'$. By our previous argument in the case in which P is even and Q is odd, we see that

$$2^k \| V_{2n-1}' = V_{3(2n-1)} \quad \text{and} \quad 2 \| V_{2n}' = V_{3(2n)}$$

for all $n \geq 1$. Thus, $2^k \| V_1$ and 2^k is a divisor of $\langle V_n \rangle$ but 2^{k+1} is not a divisor of $\langle V_n \rangle$. The result now follows.

Also solved by Paul S. Bruckman and Leonard A. G. Dresel.

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