DERIVATIVE SEQUENCES OF JACOBSTHAL AND JACOBSTHAL-LUCAS POLYNOMIALS

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1. AIM OF THE PAPER

The *Jacobsthal polynomials* $J_n(x)$ and the *Jacobsthal-Lucas polynomials* $j_n(x)$, whose properties have been investigated in [4], are a natural extension of the *Jacobsthal numbers* J_n and the *Jacobsthal-Lucas numbers* j_n which, in turn, have been investigated in [3]. These polynomials are defined by the second-order recurrence relations

$$J_{n+2}(x) = J_{n+1}(x) + 2xJ_n(x), \quad [J_0(x) = 0, \; J_1(x) = 1] \tag{1.1}$$

and

$$j_{n+2}(x) = j_{n+1}(x) + 2xj_n(x), \quad [j_0(x) = 2, \; j_1(x) = 1], \tag{1.2}$$

respectively, where x is an indeterminate.

Since throughout this paper we shall make use of the notation and the formulas found in [3] and [4], the reader is assumed to be aware of the contents of these papers.

Definitions: Following the idea exploited in [1], let us define the polynomials $J_n^{(1)}(x)$ and $j_n^{(1)}(x)$ {see (3.9) and (3.10) of [4] for the combinatorial representations of $J_n(x)$ and $j_n(x)$} as

$$J_n^{(1)}(x) = \frac{d}{dx}J_n(x) = \sum_{r=0}^{\lfloor (n-1)/2 \rfloor} 2^r r \binom{n-1-r}{r} x^{r-1} \quad (n \ge 0), \tag{1.3}$$

$$j_n^{(1)}(x) = \frac{d}{dx}j_n(x) = \sum_{r=0}^{\lfloor n/2 \rfloor} \frac{2^r nr}{n-r} \binom{n-r}{r} x^{r-1} \quad (n \ge 1), \tag{1.4}$$

and

$$J_0^{(1)}(x) = j_0^{(1)}(x) = 0, \tag{1.5}$$

where the symbol $\lfloor \cdot \rfloor$ denotes the greatest integer function, and the bracketed superscript symbolizes the first derivative with respect to x.

The aim of this paper is to study some properties of the above sequences just as was done in [1] for the Fibonacci and Lucas polynomials. Here, we shall also confine ourselves to considering the case $x = 1$. Since letting $x = 1$ in (1.1) and (1.2) will yield the Jacobsthal numbers and the Jacobsthal-Lucas numbers {cf. (2.3) and (2.4) of [3]}

$$J_n = \frac{2^n - (-1)^n}{3} \quad \text{and} \quad j_n = 2^n + (-1)^n, \tag{1.6}$$

the sequences $\{J_n^{(1)}(1)\}$ and $\{j_n^{(1)}(1)\}$ will be referred to as *Jacobsthal* and *Jacobsthal-Lucas derivative sequences*. For notational convenience, their terms $J_n^{(1)}(1)$ and $j_n^{(1)}(1)$ will be denoted by H_n and K_n, respectively. From (1.3)-(1.5), the numbers H_n and K_n can be obtained readily for the first few values of n. They are shown in Table 1.

TABLE 1. The Numbers H_n and K_n for $0 \le n \le 8$

n	0	1	2	3	4	5	6	7	8
$J_n^{(1)}(1) = H_n$	0	0	0	2	4	14	32	82	188
$j_n^{(1)}(1) = K_n$	0	0	4	6	24	50	132	294	688

2. CLOSED-FORM EXPRESSIONS FOR H_n AND K_n

Closed-form expressions for H_n and K_n are, quite obviously, useful tools for discovering their properties. They are established in this section, where some equivalent expressions for these numbers are also found.

By using formulas (1.4), (1.5), (3.3), and (3.4) of [4], we easily see that

$$\Delta^{(1)}(x) = \frac{d}{dx}\Delta(x) = 4 / \Delta(x),$$

$$\alpha^{(1)}(x) = \frac{d}{dx}\alpha(x) = \Delta^{(1)}(x)/2 = 2/\Delta(x),$$

$$\beta^{(1)}(x) = \frac{d}{dx}\beta(x) = -\Delta^{(1)}(x)/2 = -2/\Delta(x),$$

$$[\alpha^n(x)]^{(1)} = \frac{d}{dx}\alpha^n(x) = n\alpha^{n-1}(x)\alpha^{(1)}(x) = 2n\alpha^{n-1}(x)/\Delta(x),$$

and

$$[\beta^n(x)]^{(1)} = \frac{d}{dx}\beta^n(x) = n\beta^{n-1}(x)\beta^{(1)}(x) = -2n\beta^{n-1}(x)/\Delta(x).$$

Hence, we have

$$J_n^{(1)}(x) = \frac{d}{dx}\left[\frac{\alpha^n(x) - \beta^n(x)}{\Delta(x)}\right] = 2\frac{nj_{n-1}(x) - 2J_n(x)}{\Delta^2(x)}, \tag{2.1}$$

and

$$j_n^{(1)}(x) = 2nJ_{n-1}(x). \tag{2.2}$$

Letting $x = 1$ in (2.1) and (2.2) leads to the relations

$$H_n = J_n^{(1)}(1) = \frac{2(nj_{n-1} - 2J_n)}{9} \tag{2.3}$$

and

$$K_n = j_n^{(1)}(1) = 2nJ_{n-1} \tag{2.4}$$

which express H_n and K_n in terms of J_n and j_n.

By (2.3) and (2.4) above, and (1.6), the following relations can be obtained readily:

$$H_n = \frac{2^n(3n-4) - (6n-4)(-1)^n}{27} \tag{2.5}$$

and

$$K_n = \frac{n[2^n + 2(-1)^n]}{3}, \tag{2.6}$$

which express H_n and K_n in terms of their subscripts.

Observe that using (2.5) and (2.6) above, along with (1.6), we obtain the relations

$$H_n = \frac{(3n-4)J_n - n(-1)^n}{9} \tag{2.7}$$

and

$$K_n = \frac{n[j_n + (-1)^n]}{3}, \tag{2.8}$$

which express H_n in terms of J_n and K_n in terms of j_n, respectively.

3. BASIC PROPERTIES OF H_n AND K_n

Some relations involving H_n and K_n are established in this section, most of which are the analogs of those found by Horadam in [3] for J_n and j_n. Some simple but sometimes tedious manipulations involving the use of (2.3)-(2.8) provide the required proofs. To save space, only the proofs of Theorems 1-3 will be given in detail in Subsection 3.2.

3.1. Results

Generating functions

$$\sum_{n=0}^{\infty} H_n y^n = \frac{2y^3}{(2y^2 + y - 1)^2}, \tag{3.1}$$

$$\sum_{n=0}^{\infty} K_n y^n = \frac{2y^2(2-y)}{(2y^2 + y - 1)^2}. \tag{3.2}$$

These functions can be obtained readily from (3.1) and (3.2) of [4].

Recurrence relations

$$H_{n+2} = H_{n+1} + 2H_n + 2J_n, \tag{3.3}$$

$$K_{n+2} = K_{n+1} + 2K_n + 2j_n. \tag{3.4}$$

These relations can be obtained readily by calculating at $x = 1$ the first derivative with respect to x of both sides of (1.1) and (1.2).

Some identities

$$H_n K_n = \frac{n}{9}[K_{2n-1} - 2J_{n-1}(4J_n - j_{n-1})] \tag{3.5}$$

$$= \frac{n}{3}H_{2n} - \frac{n}{81}[(-2)^{n+2} + 3n4^n - 4] \tag{3.5'}$$

$$H_{n+1} + 2H_{n-1} = K_n - 2J_{n-1} \tag{3.6}$$

$$= 2(n-1)J_{n-1} \quad \text{[by (2.4)]}, \tag{3.6'}$$

$$K_{n+1} + 2K_{n-1} = 9H_n + 2J_n + 2^n, \tag{3.7}$$

$$H_n + K_n = 2H_{n+1} \quad \text{[from (2.5) and (2.6)]}, \tag{3.8}$$

$$K_n - H_n = 4(H_{n-1} + J_{n-1}), \tag{3.9}$$

$$K_n = 3H_n + \frac{1}{9}[4(-1)^n(3n-1) + 2^{n+2}]. \tag{3.10}$$

Observe that identity (3.8) is an important feature of H_n and K_n, being analogous to $J_n + j_n = 2J_{n+1}$ for Jacobsthal and Jacobsthal-Lucas numbers.

Simson formula analogs

$$H_{n+1}H_{n-1} - H_n^2 = \frac{1}{81}[(-2)^n(9n^2 - 18n + 5) - 4^n - 4], \tag{3.11}$$

$$K_{n+1}K_{n-1} - K_n^2 = -\frac{1}{9}[(-2)^n(9n^2 - 5) + 4^n + 4]. \tag{3.12}$$

Limits

$$\lim_{n\to\infty} H_{n+1} / H_n = \lim_{n\to\infty} K_{n+1} / K_n = 2, \tag{3.13}$$

$$\lim_{n\to\infty} K_n / H_n = 3. \tag{3.14}$$

Evaluation of some finite sums

$$S_n \stackrel{\text{def}}{=} \sum_{k=0}^{n} H_k = 2H_n - \frac{1}{18}[2^{n+2} - (-1)^n(6n-5) - 9], \tag{3.15}$$

$$T_n \stackrel{\text{def}}{=} \sum_{k=0}^{n} K_k = 2K_n - \frac{1}{6}[(-1)^n(6n-1) + 2^{n+2} - 3]. \tag{3.16}$$

Alternative, but perhaps less elegant, expressions for S_n and T_n can be obtained after several tedious manipulations involving the use of (2.4) and (2.7). They are

$$S_n = \frac{1}{108}[2^n(27n - 56) - 9K_n - (-1)^n(12\lceil n/2 \rceil - 5) + 51], \tag{3.15$'$}$$

where the symbol $\lceil x \rceil$ denotes the least integer not less than x, and

$$T_n = \frac{1}{2}[K_n + J_{n+1} + 2^n(n-2) + 1]. \tag{3.16$'$}$$

$$\sum_{k=0}^{n} \binom{n}{k} H_k = 2(n-2)3^{n-3} + \frac{1}{9}\left[2\delta_{1,n} + \frac{4}{3}\delta_{0,n}\right], \tag{3.17}$$

where $\delta_{a,b} = 1$ (0) if $a = (\neq) b$ is the Kronecker symbol,

$$\sum_{k=0}^{n} \binom{n}{k} K_k = 2n3^{n-2} - \frac{2}{3}\delta_{1,n}. \tag{3.18}$$

Convolution properties

$$H_n = \sum_{k=0}^{n} J_k J_{n-k} - \frac{1}{9}[2^n + (-1)^n(3n-1)], \tag{3.19}$$

$$K_n = \frac{1}{3}\sum_{k=0}^{n} j_k j_{n-k} - \frac{1}{9}[2^n 7 - (-1)^n(3n-5)], \tag{3.20}$$

$$\sum_{k=0}^{n} H_k H_{n-k} = 3^{-7}[2^{n-1}(9n^3 - 72n^2 + 159n - 80) + (-1)^n(18n^3 - 72n^2 + 30n + 40)], \quad (3.21)$$

$$\sum_{k=0}^{n} K_k K_{n-k} = 3^{-5}[2^{n-1}(9n^3 - 57n + 16) + (-1)^n(18n^3 - 42n - 8)]. \quad (3.22)$$

***Remarks*:**

(i) The geometric series formula has to be used along with (2.3)-(2.8) to prove (3.15)-(3.22).

(ii) The identities (3.19) and (3.20) can be checked easily by using (2.5), (2.6), and the identities

$$\sum_{k=0}^{n} J_k J_{n-k} = \frac{1}{9}[(n+1)j_n - 2J_{n+1}] \quad (3.23)$$

and

$$\sum_{k=0}^{n} j_k j_{n-k} = (n+1)j_n + 2J_{n+1}, \quad (3.24)$$

which are obtainable by using (1.6) and the geometric series formula.

Congruence properties

Congruence properties of H_n and K_n deserve a thorough investigation. Nevertheless, in this paper we shall confine ourselves to considering the residue of these numbers modulo their subscripts. That K_n is divisible by n for all $n > 0$ is patent by (2.4). A brief computer experiment showed that the values of $n \leq 1000$ for which H_n is divisible by n are 1, 2, 4, 20, 100, 220, 500, 620, and 820.

Theorem 1: There exist infinitely many values of n for which $H_n \equiv 0 \pmod{n}$.

Theorem 2: If $p \neq 3$ is a prime, then

$$H_p \equiv -12\left[\frac{1 + p(-1)^{p \,(\mathrm{mod}\, 3)}}{3}\right]^3 \pmod{p}.$$

Theorem 3: If $p \neq 3$ is a prime, then $K_p \equiv 0 \pmod{p^2}$.

3.2. Proofs of Special Results

Proof of Theorem 1: We shall prove that, if $n = n(k) = 5^k 4$ $(k = 0, 1, 2, \ldots)$, then $H_n \equiv 0$ (mod n). Let B_n denote the numerator of the fraction on the right-hand side of (2.5). Since $n(k)$ and 27 are coprime, it suffices to prove that $B_{n(k)} \equiv 0 \pmod{n(k)}$. After some simple manipulation, it is apparent that this is equivalent to proving that $4[2^{n(k)} - (-1)^{n(k)}] \equiv 0 \pmod{n(k)}$, that is, to proving the validity of the congruence

$$2^{n(k)} \equiv 1 \pmod{5^k}. \quad (3.25)$$

By Euler's theorem, it is known that $2^{n(k)} = 2^{5^k 4} = 2^{\phi(5^{k+1})} \equiv 1 \pmod{5^{k+1}}$, whence (3.25) is satisfied *a fortiori*.

By Table 1, it is immediately seen that the congruence $H_n \equiv 0 \pmod{n}$ holds for n = 1, 2, and 4. We now state a proposition that gives the general solution to the problem of finding *all* $n > 4$

for which this congruence is satisfied. Of course, this general solution encompasses the case $n = 5^k 4$ considered in the proof of Theorem 1.

Proposition 1: For $n > 4$, $H_n \equiv 0 \pmod{n}$ if and only if

$$n = 4p_2^{a_2} p_3^{a_3} \cdots (p_2 = 5, p_3 = 7, p_4 = 11, \ldots;\ a_2 \geq 1, a_i \geq 0 \text{ for } i > 3),$$

and $\text{ord}(2, p_i^{a_i})$ divides n for all i such that $a_i \geq 1$, where (see [2], p. 71) the symbol $\text{ord}(a, b)$ [defined for $\text{g.c.d.}(a, b) = 1$] denotes the least exponent x for which $a^x \equiv 1 \pmod{b}$.

The proof of Proposition 1 is extremely long and cumbersome; it is omitted for the sake of brevity, but it is available on request.

Proof of Theorem 2: By (2.5), we get the congruence

$$H_p \equiv \frac{-2^{p+2} - 4}{27} \equiv -\frac{12}{27} \pmod{p} \quad \text{(by Fermat's Little Theorem).}$$

The desired result is obtained readily by observing that the multiplicative inverse of 27 modulo a prime $p \neq 3$ is $\{[1 + p(-1)^{p \,(\text{mod}\, 3)}]/3\}^3$.

Proof of Theorem 3: First, by Table 1, we observe that $K_2 \equiv 0 \pmod{4}$ and $K_3 \equiv 6 \pmod{9}$. Then, for $p > 5$, let us define $M_p = K_p / p$ and prove that $M_p \equiv 0 \pmod{p}$. By (2.4) and (1.6), we can write

$$M_p = 2J_{p-1} = \frac{2^p - 2}{3} \equiv \frac{2-2}{3} \equiv 0 \pmod{p} \quad \text{(by Fermat's Little Theorem).}$$

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