ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-542 ***Proposed by H.-J. Seiffert, Berlin, Germany***

Define the sequence $(c_k)_{k\geq 1}$ by

$$c_k = \begin{cases} 1 & \text{if } k \equiv 2 \pmod 5, \\ -1 & \text{if } k \equiv 3 \pmod 5, \\ 0 & \text{otherwise.} \end{cases}$$

Show that, for all positive integers n:

$$\frac{1}{n}\sum_{k=1}^{n} k\binom{2n}{n-k}c_k = F_{2n-2}; \tag{1}$$

$$\frac{1}{2n-1}\sum_{k=1}^{2n-1}(-1)^k k\binom{4n-2}{2n-k-1}c_k = 5^{n-1}F_{2n-2}; \tag{2}$$

$$\frac{1}{2n}\sum_{k=1}^{2n}(-1)^k k\binom{4n}{2n-k}c_k = 5^{n-1}L_{2n-1}. \tag{3}$$

H-543 ***Proposed by David M. Bloom, Brooklyn College of CUNY, Brooklyn, NY***

Find all positive nonsquare integers d such that, in the continued-fraction expansion

$$\sqrt{d} = \left[n; \overline{a_1, \ldots, a_{r-1}, 2n}\right],$$

we have $a_1 = \cdots = a_{r-1} = 1$. (This includes the case $r = 1$ in which there are no a's.)

H-544 ***Proposed by Paul S. Bruckman, Highwood, IL***

Given a prime $p > 5$ such that $Z(p) = p+1$, suppose that $q = \frac{1}{2}(p^2-3)$ and $r = p^2 - p - 1$ are primes with $Z(q) = q+1$, $Z(r) = \frac{1}{2}(r-1)$. Prove that $n = pqr$ is a FPP (see previous proposals for definitions of the Z-function and of FPP's).

SOLUTIONS

Re-enter

H-525 ***Proposed by Paul S. Bruckman, Highwood, IL***
(Vol. 35, no. 1, February 1997)

Let p be any prime $\neq 2, 5$. Let

$$q = \frac{1}{2}(p-1), \quad e = \left(\frac{5}{p}\right), \quad r = \frac{1}{2}(p-e).$$

Let $Z(p)$ denote the entry-point of p in the Fibonacci sequence. Given that $2^{p-1} \equiv 1 \pmod{p}$ and $5^q \equiv e \pmod{p}$, let

$$A = \frac{1}{p}(2^{p-1}-1), \quad B = \frac{1}{p}(5^q - e), \quad C = \sum_{k=1}^{q} \frac{5^{k-1}}{2k-1}.$$

Prove that $Z(p^2) = Z(p)$ if and only if $eA - B \equiv C \pmod{p}$.

Solution by the proposer

Unless otherwise indicated, we will assume congruences (mod p), but will omit the "(mod p)" notation. Note that $(5/p) = (-1/p) = 1$. It follows from [1] that a and q have the same parity and, in fact, are both even. Since $p \equiv 1 \pmod{4}$, let $r = q/2$, an integer. Define the function $\delta_p = \delta$ as follows:

$$\delta = \begin{cases} +1 & \text{if } p \equiv 1 \pmod{20}, \\ -1 & \text{if } p \equiv 9 \pmod{20}. \end{cases} \tag{1}$$

We may therefore express the desired result as follows:

$$\delta \cdot 5^r \equiv (-1)^{a/2+r}. \tag{2}$$

The following result was shown in [2]:

$$F_{q+1} \equiv (-1)^{a/2+r}. \tag{3}$$

Also, note that $(\alpha\beta/p) = (-1/p) = 1$, hence $(\alpha/p) = (\beta/p)$; note that since $(5/p) = 1$, $\sqrt{5}$ and, hence, α and β are ordinary residues. Then,

$$F_{q+1} = 5^{-1/2}(\alpha^{q+1} - \beta^{q+1}) = 5^{-1/2}(\alpha^q\alpha - \beta^q\beta) \equiv (\alpha-\beta)^{-1}\{(\alpha/p)\alpha - (\beta/p)\beta\},$$

or

$$F_{q+1} \equiv (\alpha/p). \tag{4}$$

In light of (2), (3), and (4), it suffices to prove that

$$(\alpha/p) \equiv \delta \cdot 5^r. \tag{5}$$

Note that $5^r = (\sqrt{5})^q \equiv (\sqrt{5}/p)$. Therefore, it suffices to prove that

$$(\alpha/p) = \delta(\sqrt{5}/p). \tag{6}$$

However, the last result is an old result attributable to E. Lehmer (see [3]); we have only changed the notation to conform with that employed herein. Thus, the desired result is established.

References

1. D. M. Bloom. Problem H-494. *The Fibonacci Quarterly* **33.1** (1995):91. The solution by H.-J. Seiffert appeared in *The Fibonacci Quarterly* **34.2** (1996):190-91.
2. P. S. Bruckman. Problem H-515. *The Fibonacci Quarterly* **34.4** (1996):379.
3. E. Lehmer. "On the Quadratic Character of the Fibonacci Root." *The Fibonacci Quarterly* **4.2** (1966):135-38.

Also solved by H.-J. Seiffert.

Generator Trouble

<u>H-526</u> ***Proposed by Paul S. Bruckman, Highwood, IL***
(Vol. 35, no. 2, May 1997)

Following H-465, let $r_1, r_2,$ and r_3 be natural integers such that

(1) $\sum_{k=1}^{3} kr_k = n$, where n is a given natural integer.

Let

(2) $B_{r_1,r_2,r_3} = \dfrac{1}{r_1+r_2+r_3}\dfrac{(r_1+r_2+r_3)!}{r_1!r_2!r_3!}.$

Also, let

(3) $C_n = \sum B_{r_1,r_2,r_3}$, summed over all possible $r_1, r_2,$ and r_3.

Define the generating function

(4) $$F(x) = \sum_{n=6}^{\infty} C_n x^n:$$

(a) find a closed form for $F(x)$;
(b) obtain an explicit expression for C_n;
(c) show that C_n is a positive integer for all $n \geq 7$, n prime.

Solution by the proposer

Solution of part (a): Note that $2 \leq 2r_2 \leq n-1-3r_3 \leq n-4$ (eliminating $r_1 = n-2r_2-3r_3$). Then

$$F(x) = \sum_{n=6}^{\infty} x^n \sum_{r_3=1}^{[n/3-1]^3} 1/r_3! \sum_{r_2=1}^{\left[\frac{1}{2}(n-1-3r_3)\right]} \frac{(n-2r_3-r_2-1)!}{r_2!(n-3r_3-2r_2)!}$$
$$= \sum_{r_3=1}^{\infty} \frac{1}{r_3!} \sum_{n=3r_3+3}^{\infty} x^n \sum_{r_2=1}^{\left[\frac{1}{2}(n-1-3r_3)\right]} \frac{(n-2r_3-r_2-1)!}{r_2!(n-3r_3-2r_2)!}.$$

Changing variables, we obtain

$$F(x) = \sum_{v=1}^{\infty} \frac{1}{v!} \sum_{m=0}^{\infty} x^{m+3v+3} \sum_{u=1}^{\left[\frac{1}{2}(m+2)\right]} \frac{(m+2+v-u)!}{u!(m+3-2u)!}$$
$$= \sum_{m=0}^{\infty} x^{m+1} \sum_{u,\,v=1}^{\infty} x^{2u+3v} \frac{(m+u+v)!}{(m+1)!\,u!\,v!} = \sum_{n=1}^{\infty} \frac{x^n}{n} \sum_{v=1}^{\infty} x^{3v} \binom{n-1+v}{v} \sum_{u=1}^{\infty} x^{2u} \binom{n+v-1+u}{u}$$

$$=\sum_{n=1}^{\infty}\frac{x^n}{n}\sum_{v=1}^{\infty}x^{3v}\binom{n-1+v}{v}\cdot[(1-x^2)^{-n-v}-1]$$

$$=\sum_{n=1}^{\infty}\frac{x^n}{n}\sum_{v=1}^{\infty}(-x^3)^v\binom{-n}{v}[(1-x^2)^{-n-v}-1]$$

$$=\sum_{n=1}^{\infty}\frac{x^n}{n}\left\{(1-x^2)^{-n}\left[\left(1-\frac{x^3}{1-x^2}\right)^{-n}-1\right]-[(1-x^3)^{-n}-1]\right\}$$

$$=\sum_{n=1}^{\infty}\frac{1}{n}\left[\left(\frac{x}{1-x^2-x^3}\right)^n-\left(\frac{x}{1-x^2}\right)^n-\left(\frac{x}{1-x^3}\right)^n+x^n\right]$$

$$=-\log\left(1-\frac{x}{1-x^2-x^3}\right)+\log\left(1-\frac{x}{1-x^2}\right)+\log\left(1-\frac{x}{1-x^3}\right)-\log(1-x)$$

$$=-\log(1-x-x^2-x^3)+\log(1-x^2-x^3)+\log(1-x-x^2)-\log(1-x^2)$$
$$+\log(1-x-x^3)-\log(1-x^3)-\log(1-x),$$

or

$$F(x)=\log\left\{\frac{(1-x-x^2)(1-x^2-x^3)(1-x-x^3)}{(1-x)(1-x^2)(1-x^3)(1-x-x^2-x^3)}\right\}. \qquad (*)$$

Solution of part (b): Suppose

$$\begin{aligned}1-x^2-x^3&=(1-rx)(1-sx)(1-tx),\\ 1-x-x^3&=(1-ux)(1-vx)(1-wx),\\ 1-x-x^2-x^3&=(1-fx)(1-gx)(1-hx).\end{aligned} \qquad (**)$$

Then

$$\begin{aligned}F(x)=&\log(1-\alpha x)+\log(1-\beta x)+\log(1-rx)+\log(1-sx)+\log(1-tx)\\ &+\log(1-ux)+\log(1-vx)+\log(1-wx)-3\log(1-x)-\log(1+x)\\ &-\log(1-\omega x)-\log(1-\omega^2x)-\log(1-fx)-\log(1-gx)-\log(1-hx),\end{aligned}$$

where α and β are the usual Fibonacci constants and $\omega=\exp(2i\pi/3)$. We then obtain

$$\begin{aligned}F(x)=\sum_{n=1}^{\infty}\frac{x^n}{n}[&-(\alpha^n+\beta^n)-(r^n+s^n+t^n)-(u^n+v^n+w^n)\\ &+3+(-1)^n+\omega^n+\omega^{2n}+(f^n+g^n+h^n)].\end{aligned}$$

Comparison of coefficients yields the explicit formula:

$$C_n=\frac{1}{n}(J_n+3+(-1)^n+\omega^n+\omega^{2n}-L_n-G_n-H_n),\quad n=1,2,3,\ldots, \qquad (***)$$

where

$$L_n=\alpha^n+\beta^n \text{ (Lucas numbers)},\quad G_n=r^n+s^n+t^n,$$
$$H_n=u^n+v^n+w^n,\quad J_n=f^n+g^n+h^n,\quad n=1,2,\ldots. \qquad (****)$$

The initial values and recurrence relations satisfied by the J_n's, G_n's, and H_n's may be obtained from $(**)$, and are as follows:

(i) $J_{n+3} = J_{n+2} + J_{n+1} + J_n,\ n = 1, 2, \ldots;\ J_1 = 1, J_2 = 3, J_3 = 7;$

(ii) $G_{n+3} = G_{n+1} + G_n,\ n = 1, 2, \ldots;\ G_1 = 0, G_2 = 2, G_3 = 3;$

(iii) $H_{n+3} = H_{n+2} + H_n,\ n = 1, 2, \ldots;\ H_1 = H_2 = 1, H_3 = 4.$

If $n \geq 5$ is prime, $\omega^n + \omega^{2n} = -1$; thus, for prime $n \geq 7$, we obtain the slightly simplified formula for C_n:

$$C_n = \frac{1}{n}(J_n + 1 - L_n - G_n - H_n),\ n \geq 7,\ n \text{ prime.} \qquad (*****)$$

To obtain values of J_n, G_n, and H_n without means of the recurrence relations (i)-(iii), we would need to solve for the roots in $(**)$; we shall omit this exercise and assume that these roots are known. Also, it is of interest to note, as can be verified, that C_n given by $(***)$ vanishes for $n = 1, 2, 3, 4, 5$, as we would expect.

Solution of part (c): As was determined in Problem H-465 as a special case, B_{r_1, r_2, r_3} is an integer for prime $n \geq 7$. From (3), it then follows immediately that C_n is an integer if n is prime (even for $n = 2, 3, 5$, since $C_2 = C_3 = C_5 = 0$.)

Note: It may be shown that $L_n \equiv 1 \pmod{n}$ for all prime n; from this result and the expression in $(*****)$, we deduce that

$$J_n \equiv G_n + H_n \pmod{n}, \text{ if } n \text{ is prime.} \qquad (\#)$$

Sum Formula

H-527 ***Proposed by N. Gauthier, Royal Military College of Canada (Vol. 35, no. 2, May 1997)***

Let q, a, and b be positive integers, with $(a, b) = 1$. Prove or disprove the following:

a)
$$\sum_{\substack{r=0\\(br+as<ab)}}^{a-1}\sum_{s=0}^{b-1}(-1)^{q(br+as)}L_{2q(br+as)} = \frac{F_{q(a+b-ab)}F_{qab}}{F_{qa}F_{qb}} + (-1)^{q(1-ab)}\frac{F_{q(2ab-1)}}{F_q};$$

b)
$$5\sum_{\substack{r=0\\(br+as<ab)}}^{a-1}\sum_{s=0}^{b-1}(-1)^{q(br+as)}L_{2q(br+as)} = (-1)^{q(1-ab)}\frac{L_{q(2ab-1)}}{F_q} - \frac{F_{qab}L_{q(a+b-ab)}}{F_{qa}F_{qb}}.$$

Solution by the proposer

Consider

$$S(x; a, b) \equiv \sum_{\substack{r=0\\(br+as<ab)}}^{a-1}\sum_{s=0}^{b-1} x^{br+as}, \qquad (1)$$

for a, b positive integers, with $(a, b) = 1$, and $x \neq 1$ an arbitrary variable. L. Carlitz has shown ["Some Restricted Multiple Sums," *The Fibonacci Quarterly* **18.1** (1980):58-65, eqns. (1.1) and (1.2)] that

$$S(x; a, b) = \frac{1 - x^{ab}}{(1 - x^a)(1 - x^b)} - \frac{x^{ab}}{1 - x}. \qquad (2)$$

Now, for q a positive integer, consider

$$T_{\pm}(q;a,b) \equiv S(\alpha^q/\beta^q;a,b) \pm S(\beta^q/\alpha^q;a,b), \tag{3}$$

where $\alpha = \frac{1}{2}[a+\sqrt{5}]$, $\beta = \frac{1}{2}[1-\sqrt{5}]$, $\alpha\beta = -1$. It is readily seen that (2) in (3) gives

$$\begin{aligned}\sqrt{5}\,T_{\pm} &= -\beta^{q(a+b-ab)}\frac{F_{qab}}{F_{qa}F_{qb}} + \alpha^{qab}\beta^{q(1-ab)}\frac{1}{F_q} \\ &\pm\left[\alpha^{q(a+b-ab)}\frac{F_{qab}}{F_{qa}F_{qb}} - \beta^{qab}\alpha^{q(1-ab)}\frac{1}{F_q}\right],\end{aligned} \tag{4}$$

where $F_n \equiv (\alpha^n - \beta^n)/\sqrt{5}$. Similarly, (1) in (3) gives

$$\begin{aligned}\sqrt{5}\,T_{\pm} &= \sqrt{5}\sum_{\substack{r=0\\(br+as<ab)}}^{a-1}\sum_{s=0}^{b-1}\left[\left(\frac{\alpha^q}{\beta^q}\right)^{br+as} \pm \left(\frac{\beta^q}{\alpha^q}\right)^{br+as}\right] \\ &= \sqrt{5}\sum_{\substack{r=0\\(br+as<ab)}}^{a-1}\sum_{s=0}^{b-1}(-1)^{q(br+as)}\left[\alpha^{2q(br+as)} \pm \beta^{2q(br+as)}\right].\end{aligned} \tag{5}$$

The solution to part (a) follows by choosing T_+ in (4) and (5); equating the results gives

$$\sum_{\substack{r=0\\(br+as<ab)}}^{a-1}\sum_{s=0}^{b-1}(-1)^{q(br+as)}L_{2q(br+as)} = \frac{F_{q(a+b-ab)}F_{qab}}{F_{qa}F_{qb}} + (-1)^{q(1-ab)}\frac{F_{q(2ab-1)}}{F_q}.$$

For the solution to part (b), choose T_- in (4) and (5) to obtain

$$5\sum_{\substack{r=0\\(br+as<ab)}}^{a-1}\sum_{s=0}^{b-1}(-1)^{q(br+as)}L_{2q(br+as)} = (-1)^{q(1-ab)}\frac{L_{q(2ab-1)}}{F_q} - \frac{F_{qab}L_{q(a+b-ab)}}{F_{qa}F_{qb}}.$$

Also solved by P. Bruckman.

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