ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by Stanley Rabinowitz

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to stan@mathpro.com *on Internet. All correspondence will be acknowledged.*

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions. Although this Elementary Problem section does not insist on original problems, we do ask that proposers inform us of the history of the problem, if it is not original. A problem should not be submitted elsewhere while it is under consideration for publication in this column.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \ F_0 = 0, \ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n, \ L_0 = 2, \ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-854 ***Proposed by Paul S. Bruckman, Edmonds, WA***

Simplify

$$3\arctan(\alpha^{-1}) - \arctan(\alpha^{-5}).$$

B-855 ***Proposed by the editor***

Let $r_n = F_{n+1}/F_n$ for $n > 0$. Find a recurrence for r_n.

B-856 ***Proposed by Zdravko F. Starc, Vršac. Yugoslavia***

If n is a positive integer, prove that

$$L_1\sqrt{F_1} + L_2\sqrt{F_2} + L_3\sqrt{F_3} + \cdots + L_n\sqrt{F_n} < 8F_n^2 + 4F_n.$$

B-857 ***Proposed by the editor***

Find a sequence of integers $\langle w_n \rangle$ satisfying a recurrence of the form $w_{n+2} = Pw_{n+1} - Qw_n$ for $n \ge 0$, such that for all $n > 0$, w_n has precisely n digits (in base 10).

B-858 ***Proposed by Wolfdieter Lang, Universität Karlsruhe, Germany***

(a) Find an explicit formula for

$$\sum_{k=0}^{n} kF_{n-k}$$

which is the convolution of the sequence $\langle n \rangle$ and the sequence $\langle F_n \rangle$.

(b) Find explicit formulas for other interesting convolutions.

(The convolution of the sequence $\langle a_n \rangle$ and $\langle b_n \rangle$ is the sum $\sum_{k=0}^{n} a_k b_{n-k}$.)

B-859 ***Proposed by Kenneth B. Davenport, Pittsburgh, PA***

Simplify

$$\begin{vmatrix} F_nF_{n+1} & F_{n+1}F_{n+2} & F_{n+2}F_{n+3} \\ F_{n+3}F_{n+4} & F_{n+4}F_{n+5} & F_{n+5}F_{n+6} \\ F_{n+6}F_{n+7} & F_{n+7}F_{n+8} & F_{n+8}F_{n+9} \end{vmatrix}.$$

NOTE: The Elementary Problems Column is in need of more **easy**, yet elegant and nonroutine problems.

SOLUTIONS

Radical Inequality

B-834 ***Proposed by Zdravko F. Starc, Vršac, Yugoslavia (Vol. 35, no. 3, August 1997)***

For x a real number and n an integer larger than 1, prove that

$$(x+1)F_1 + (x+2)F_2 + \cdots + (x+n)F_n < 2^n\sqrt{\frac{n(n+1)(2n+1+6x)+6nx^2}{6}}.$$

Editorial comment: *In the original statement of the problem in the August issue, the final term "$6nx^2$" was erroneously printed as "nx^2". Nevertheless, Bruckman managed to prove the stronger inequality, as it was actually printed. We now present his solution to this stronger inequality.*

Solution 2 by Paul S. Bruckman, Edmonds, WA

The inequality is false if the radicand is negative, so to simplify matters, we impose the condition $x \geq 0$, which ensures that the right member is well defined.

Let

$$S(x,n) = \sum_{k=1}^{n}(x+k)F_k$$

denote the left member of the putative inequality. Then

$$\begin{aligned} S(x,n) &= \sum_{k=1}^{n}[(x+k)F_{k+2} - F_{k+3} - (x+k-1)F_{k+1} + F_{k+2}] \\ &= (x+n)F_{n+2} - F_{n+3} - xF_2 + F_3 \\ &= (x+n)F_{n+2} - F_{n+3} - (x-2) \\ &= x(F_{n+2}-1) + (n-1)F_{n+2} - F_{n+1} + 2. \end{aligned}$$

Since $n \geq 2$, we have $n \leq 2^{n-1}$, $F_{n+1} \geq 2$, and $F_{n+2} \leq 1 + 2^{n-1}$. Therefore,

$$S(x,n) < x \cdot 2^{n-1} + n(1+2^{n-1}) \leq 2^{n-1}(x+n+1).$$

If

$$R(x,n)=2^n\sqrt{\frac{n(n+1)(2n+1+6x)+nx^2}{6}}$$

represents the right member of the putative inequality, then

$$\begin{aligned}R(x,n)&=2^n\sqrt{n/6}\cdot\sqrt{x^2+6(n+1)x+(n+1)(2n+1)}\\&>2^{n-1}\sqrt{x^2+2(n+1)x+(n+1)^2}\\&=2^{n-1}(x+n+1).\end{aligned}$$

Thus, $S(x,n)<R(x,n)$ if $n\geq 2$.

Weighted Binomial Sum

B-839 ***Proposed by David M. Bloom, Brooklyn College of CUNY, Brooklyn, NY (Vol. 35, no. 4, November 1997)***

Evaluate the sum

$$\sum_{k=0}^{\lfloor n/3\rfloor}(-1)^k 2^{-3k}\binom{n-2k}{k}$$

in terms of Fibonacci numbers.

Comment: Seiffert and Prielipp pointed out that the result

$$\sum_{k=0}^{\lfloor n/3\rfloor}(-1)^k 2^{-3k}\binom{n-2k}{k}=\frac{F_{n+3}-1}{2^n}$$

was proven by Jaiswal in [1].

Reference

1. D. V. Jaiswal. "On Polynomials Related to the Tchebichef Polynomials of the Second Kind." *The Fibonacci Quarterly* **12.3** (1974):263-65.

Seiffert found several other related identities, such as the remarkable formula

$$\sum_{k=0}^{\lfloor n/3\rfloor}(-1)^k 16^k 17^{-3k}\binom{n-2k}{k}=\frac{2^{2n+1}F_{3n+6}+2^{2n+3}F_{3n+3}-1}{31\cdot 17^n}.$$

Using the Binet form for the Fibonacci polynomials, he was able to show that

$$\sum_{k=0}^{\lfloor n/3\rfloor}(-1)^k (x^2)^k (x^2+1)^{-3k}\binom{n-2k}{k}=\frac{x^{n+1}F_{n+2}(x)+x^{n+2}F_{n+1}(x)-1}{(x^2+1)^n(2x^2-1)},$$

where $F_{n+2}(x)=xF_{n+1}(x)+F_n(x)$ *with* $F_0(x)=0$ *and* $F_1(x)=1$.

Rabinowitz looked at the sum

$$x_n=2^n\sum_{k=0}^{\lfloor n/3\rfloor}2^{-3k}\binom{n-2k}{k}$$

and found that it satisfies the recurrence $x_n=2x_{n-1}+x_{n-3}$, *but he found no further generalization.*

Solutions also received from Paul S. Bruckman, Nenad Cakic, Charles K. Cook, Russell Jay Hendel, H.-J. Seiffert, and the proposer.

An Arcane Formula for a Curious Matrix

B-840 ***Proposed by the editor***
(Vol. 35, no. 4, November 1997)

Let

$$A_n = \begin{pmatrix} F_n & L_n \\ L_n & F_n \end{pmatrix}.$$

Find a formula for A_{2n} in terms of A_n and A_{n+1}.

Solution by Paul S. Bruckman, Edmonds, WA

We require the following identities:

$$F_{n-1}F_n + F_nF_{n+1} = F_{2n}; \quad F_{n-1}L_n + F_nL_{n+1} = L_{2n}.$$

The first identity is obvious, from the relations $F_{n-1}+F_{n+1} = L_n$ and $F_nL_n = F_{2n}$. Since $F_{n-1}L_n = F_{2n-1} + (-1)^n$ and $F_nL_{n+1} = F_{2n+1} - (-1)^n$, we see that the left side of the second identity is equal to $F_{2n-1} + F_{2n+1} = L_{2n}$, as claimed.

From these identities, it follows immediately that

$$A_{2n} = F_{n-1}A_n + F_nA_{n+1}.$$

Comment by the editor: All solvers came up with the formula $A_{2n} = F_{n-1}A_n + F_nA_{n+1}$, but this formula expresses A_{2n} in terms of A_n, A_{n+1}, F_n, and F_{n-1}. What the proposer wanted, and perhaps did not state clearly enough, was a formula for A_{2n} expressed in terms of A_n and A_{n+1} **only**. The proposer's solution was

$$A_{2n} = \frac{1}{4}\left[\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} A_n^2 - \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} A_nA_{n+1} + \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} A_{n+1}^2\right].$$

How the proposer found this arcane formula remains a mystery.

Editorial comment: *In general, we have* $H_{2n} = F_{n-1}H_n + F_nH_{n+1}$ *for any sequence satisfying the recurrence* $H_n = H_{n-1} + H_{n-2}$.

Solutions also received by Brian D. Beasley, Charles K. Cook, Leonard A. G. Dresel, Russell Jay Hendel, Carl Libis, H.-J. Seiffert, and the proposer.

Integer Quotient

B-841 ***Proposed by David Zeitlin, Minneapolis, MN***
(Vol. 35, no. 4, November 1997)

Let P be an integer. For $n \geq 0$, let $U_{n+2} = PU_{n+1} + U_n$, with $U_0 = 0$ and $U_1 = 1$. Let $V_{n+2} = PV_{n+1} + V_n$, with $V_0 = 2$ and $V_1 = P$. Prove that

$$\frac{V_n^2 + V_{n+a}^2}{U_n^2 + U_{n+a}^2}$$

is always an integer if a is odd.

Solution by Hai-Xing Zhao, Qinghai Normal University, Xining, QingHai, China

From [1], we have

$$U_n = \frac{\alpha^n - \beta^n}{\sqrt{\Delta}} \quad \text{and} \quad V_n = \alpha^n + \beta^n,$$

where $\alpha + \beta = P$, $\alpha\beta = -1$, and $\Delta = P^2 + 4$. Hence,

$$U_n^2 = \frac{1}{\Delta}(V_{2n} - 2(-1)^n) \quad \text{and} \quad V_n^2 = V_{2n} + 2(-1)^n.$$

Therefore,

$$\frac{V_n^2 + V_{n+a}^2}{U_n^2 + U_{n+a}^2} = \Delta \frac{V_{2n} + V_{2n+2a} + 2(-1)^n + 2(-1)^{n+a}}{V_{2n} + V_{2n+2a} - 2(-1)^n - 2(-1)^{n+a}}.$$

When a is odd,

$$\frac{V_n^2 + V_{n+a}^2}{U_n^2 + U_{n+a}^2} = \Delta = P^2 + 4$$

which is an integer.

Reference

1. M. N. S. Swamy. "On a Class of Generalized Polynomials." *The Fibonacci Quarterly* **35.4** (1997):329-40.

Solutions also received by Paul S. Bruckman, Leonard A. G. Dresel, Herta T. Freitag, Russell Jay Hendel, H.-J. Seiffert, and the proposer.

Divisibility by $x - 1$

B-842 ***Proposed by the editor***
(Vol. 36, no. 1, February 1998)

The Fibonacci polynomials, $F_n(x)$, and the Lucas polynomials, $L_n(x)$, satisfy

$$F_{n+2}(x) = xF_{n+1}(x) + F_n(x), \quad F_0(x) = 0, \quad F_1(x) = 1;$$
$$L_{n+2}(x) = xL_{n+1}(x) + L_n(x), \quad L_0(x) = 2, \quad L_1(x) = x.$$

Prove that no Lucas polynomial is exactly divisible by $x - 1$.

Solution by Lawrence Somer, The Catholic University of America, Washington, D.C.

We prove, more generally, that if a is a nonzero real number, then no Lucas polynomial is exactly divisible by $x - a$ for $n \geq 0$ and no Fibonacci polynomial is exactly divisible by $x - a$ for $n \geq 1$.

By the Factor Theorem, $x - a$ exactly divides $L_n(x)$ if and only if $L_n(a) = 0$ and $x - a$ exactly divides $F_n(x)$ if and only if $F_n(a) = 0$. Clearly, $L_n(a)$ and $F_n(a)$ both satisfy the second-order recurrence

$$W_{n+2}(a) = aW_{n+1}(a) + W_n(a)$$

with initial terms $L_0(a) = 2$, $L_1(a) = a$, and $F_0(a) = 0$, $F_1(a) = 1$. It is easily proved by induction that $L_n(-a) = (-1)^n L_n(a)$ and $F_n(-a) = (-1)^{n+1} F_n(a)$. Thus,

$$|L_n(-a)| = |L_n(a)| \quad \text{and} \quad |F_n(-a)| = |F_n(a)|.$$

It thus suffices to prove that if $a > 0$, then $L_n(a) > 0$ for $n \geq 0$ and $F_n(a) > 0$ for $n \geq 1$. These assertions easily follow by induction. We are now done.

Solutions also received by Paul S. Bruckman, Leonard A. G. Dresel, Russell Euler & Jawad Sadek (jointly), Russell Jay Hendel, Harris Kwong, Angel Plaza & Miguel A. Pedrón (jointly), H.-J. Seiffert, Indulis Strazdins, and the proposer.

Addenda: The editor wishes to apologize for misplacing some solutions that were sent in on time. We therefore acknowledge solutions from the following solvers.

Peter G. Anderson—B-814
Brian Beasley—B-836, 837, 838
Leonard Dresel—B-784 to 789; 814, 815, 816, 819
Frank Flanigan—B-815
Pentti Haukkanen—B-837
Russell Hendel—B-814, 815, 816, 819, 821, 822, 823
Daina Krigens—B-814
Carl Libis—B-836, 837, 838
Graham Lord—B-815, 816
Bob Prielipp—B-836
Don Redmond—B-795
Lawrence Somer—B-796 to 801, 814

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