A NOTE ON THE SET OF ALMOST-ISOSCELES RIGHT-ANGLED TRIANGLES

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1. INTRODUCTION

Pythagorean triples have long provided a source of great interest and amusement to mathematicians since antiquity. Not surprisingly, many important properties have and continue to be deduced about these solutions of the diophantine equation

$$x^2 + y^2 = z^2. \tag{1}$$

The most general solutions of (1) that satisfy the conditions (see [2], p. 190) $x > 0$, $y > 0$, $z > 0$, $(x, y) = 1$, $2|x$, are

$$x = 2ab, \ y = b^2 - a^2, \ z = a^2 + b^2, \tag{2}$$

where the integers a, b are of opposite parity and $(a, b) = 1$, $b > a > 0$.

One important class of Pythagorean triples (x, y, z) that we shall be concerned with are those in which x and y are consecutive integers. These triples, which we shall call *almost-isosceles right-angled* (AIRA) triangles, can be constructed from (2) in the following manner (see [4], p. 13). Take an a and b that generate a triangle whose two shortest sides differ by one, then the next such triangle is constructed by b and $a + 2b$. Thus, as (3, 4, 5) is generated by (2) using $a = 1$ and $b = 2$, so the next AIRA triangle will be determined via $a = 2$ and $b = 5$, thereby producing the triple $(20, 21, 29)$. Clearly, with repeated applications of the rule $(a, b) \mapsto (b, a + 2b)$, one can generate an infinite number of these triples. A similar recurrence scheme generating AIRA triangles was also developed in [1] using Pell's equation.

Our aim in this short note is to re-establish the existence of infinitely AIRA triangles via an alternate argument which, unlike the above, does not require the use of (2) or Pell's equation. We shall, as a result of this approach, reveal a surprising connection that exists between these Pythagorean triples and the set of square triangular numbers [see (3)]. This will be employed later to calculate the first six such triples. An additional number fact concerning the primes is also deduced.

2. MAIN CONSTRUCTION

Let us begin by noting the following key observation, a proof is included for completeness.

Lemma 2.1: There are infinitely many perfect squares of the form $n(n+1)/2$.

Proof: If $n \in \mathbb{Z}^+$ is such that $T(n) = n(n+1)/2$ is a perfect square, then so is $T(4n(n+1))$; however, the statement now readily follows because $T(1) = 1$ is clearly a perfect square. □

To find all AIRA triangles, we first reduce the problem (as in [1]) to a question of the solubility of a diophantine equation obtained by exploiting an obvious fact, namely, if the sum of two consecutive squares is a perfect square, it must be the square of an odd number. Then, using a series of elementary arguments, one can further reduce this equation to another diophantine equation which is known to be solvable from Lemma 2.1. This approach is contained in the proof of the following theorem.

Theorem 2.1: There are infinitely many nontrivial pairs of consecutive squares whose sum is a perfect square; moreover, all such AIRA triangles are given by

$$\left(\frac{4T_n - 1 + \sqrt{8T_n^2+1}}{2}, \frac{4T_n + 1 + \sqrt{8T_n^2+1}}{2}, 2T_n + \sqrt{8T_n^2+1}\right), \tag{3}$$

where T_n denotes the positive square root of the n^{th} square triangular number.

Proof: Suppose $m \in \mathbb{Z}^+$ is such that the sum of m^2 and $(m+1)^2$ is a square, then there must exist an $s \in \mathbb{Z}^+$ such that $m^2 + (m+1)^2 = (2s+1)^2$. Upon expanding and simplifying, we obtain

$$m(m+1) = 2s(s+1). \tag{4}$$

Our argument is therefore reduced to demonstrating the infinitude of solutions (m, s) to the above diophantine equation. To solve (4), first observe that if $m \le s$ then $m(m+1) < 2s(s+1)$, while if $m \ge 2s$ then $m(m+1) > 2s(s+1)$. Thus, for an arbitrary $s \in \mathbb{Z}^+$, the only integer values m can assume in order that (4) may possibly be satisfied are those in which $2s > m > s$. Hence, if (m, s) is a solution, then there must exist a fixed $r \in \mathbb{Z}^+$ such that $m = s + r$ with

$$(s+r)(s+r+1) = 2s(s+1). \tag{5}$$

Expanding and simplifying (5) yields the quadratic, $s^2 + s(1-2r) - (r^2 + r) = 0$, in the variable s, from which it is deduced that

$$s = \frac{2r - 1 + \sqrt{8r^2+1}}{2}. \tag{6}$$

Note that the positive radical has been taken as $s > 0$. Since s is an integer $8r^2 + 1$ must be an odd perfect square. Consequently, we require that $8r^2 + 1 = (2n+1)^2$ for some $n \in \mathbb{Z}^+$, and so r and n are solutions of the diophantine equation $2r^2 = n(n+1)$.

However, by Lemma 2.1, there are infinitely many integer solutions of this equation; hence, we conclude that there are an infinite number of integers s of the form in (6) such that (5) is satisfied. Thus, equation (4) must have infinitely many solutions (m, s) since $m = s + r$.

It is now a simple matter to determine the required expression of the AIRA triples

$$(m, m+1, 2s+1). \tag{7}$$

Let T_n denote the positive square root of the n^{th} square triangular number, then by the above, $r = T_n$ and so, from (6), we have

$$s = \frac{2T_n - 1 + \sqrt{8T_n^2+1}}{2}.$$

Finally, substituting the corresponding expressions for $m = s + r$ and $m + 1$ into (7) produces (3). □

We now prove an interesting fact concerning the prime numbers, which can be deduced by showing that the odd base length in the above AIRA triangles is always a composite number, with the exception of $(3, 4, 5)$.

Corollary 2.1: If p is a prime number greater than 3, then neither $(p-1)^2+p^2$ nor $p^2+(p+1)^2$ is a perfect square.

Proof: Let $m, s \in \mathbb{Z}^+$ be such that $(m, m+1, 2s+1)$ is an AIRA triangle, and assume m is a prime greater than 3. Clearly, the integers satisfy equation (4) of the previous theorem and so $2s > m > s$. But, in view of this inequality and by assumption, we must have $(m, s) = (m, 2) = 1$; this, in turn, implies $(m, 2s) = 1$. Further, note that $m > s+1$, for if $m = s+1 = s+T_1$, then

$$m = \frac{4T_1 - 1 + \sqrt{8T_1^2+1}}{2} = 3,$$

which is contrary to the assumption. Thus, $(m, s+1) = 1$ because m is prime; consequently, $(m, 2s(s+1)) = 1$ and so m is not a divisor of $2s(s+1)$. This is a contradiction, since $m|2s(s+1)$ by equation (4). Hence, m cannot be a prime greater than 3. Assume now that $m+1$ is a prime greater than 3. Clearly, from the above inequality, $m+1 > s+1 > s$, which, via the assumption, implies that $(m+1, s+1) = (m+1, s) = (m+1, 2) = 1$. Thus, $(m+1, 2s(s+1)) = 1$, and so $m+1$ is not a divisor of $2s(s+1)$, again a contradiction because $m+1|2s(s+1)$. Consequently, since both m and $m+1$ are greater than 3 in all Pythagorean triples of the form $(m, m+1, 2s+1)$, with the exception of $(3, 4, 5)$, we deduce that m and $m+1$ must be composite. The result now readily follows. □

In view of the previous result, one may question whether the hypotenuses in the above AIRA triples are similarly composite for large n. This is the motivation behind the following conjecture.

Conjecture 2.1: There are only finitely many primes p, such that p^2 is representable as a sum of two consecutive squares.

At present the author, via an application of Corollary 5.14 in [3], has found that primes of the form $4k+3$ will fail the condition of the conjecture. Whether there exists an infinite subset of primes of the form $4k+1$ satisfying the above, is still an open question.

3. NUMERICAL COMPUTATION

To conclude this note, we shall apply equation (2.1) to calculate the first six triples. Clearly, all that is required is a means of determining T_n; however, we are fortunate in this respect, as one may either make use of the formula (see [4], p. 16)

$$T_n^2 = \frac{1}{32}((17+12\sqrt{2})^n + (17-12\sqrt{2})^n - 2), \tag{8}$$

or the recurrence relation where, for all integers $n \geq 2$,

$$T_{n+1} = 6T_n - T_{n-1}, \tag{9}$$

with $T_1 = 1$, $T_2 = 6$. Curiously, equation (8) coupled with (3) will produce an explicit formula in n for the n^{th} AIRA triangle. However, from a computational viewpoint, it is more efficient to use

(9) because this provides a recurrence scheme for calculating all AIRA triangles entailing fewer arithmetic operations. The results of the first six iterations are tabulated as follows:

TABLE 1. The First Six AIRA Triangles

n	T_n	$(4T_n - 1 + (8T_n^2 + 1)^{1/2}) / 2$	$(4T_n + 1 + (8T_n^2 + 1)^{1/2}) / 2$	$2T_n + (8T_n^2 + 1)^{1/2}$
1	1	3	4	5
2	6	20	21	29
3	35	119	120	169
4	204	696	697	985
5	1189	4059	4060	5741
6	6930	23660	23661	33461

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