COMPLETE PARTITIONS

SeungKyung Park*
Department of Mathematics, Yonsei University, Seoul 120-749, Korea
(*Submitted November 1996*)

1. INTRODUCTION

MacMahon [4] introduced perfect partitions of a number. He defined a perfect partition of a positive integer n to be a partition such that every number from 1 to n can be represented by the sum of parts of the partition in one and only one way. For instance, $(1\ 1\ 1\ 4)$ is a perfect partition of 7 because we can express each of the numbers 1 through 7 uniquely by using the parts of three 1's and one 4; thus, $(1), (1\ 1), (1\ 1\ 1), (4), (1\ 4), (1\ 1\ 4)$, and $(1\ 1\ 1\ 4)$ are the partitions referred to in this example. MacMahon considered the case of $n = p^{\alpha} - 1$, where p is a prime number, and showed that the enumeration of perfect partitions is identical to the enumeration of compositions of the number α, using the correspondence between the factorizations of $(1 - x^{p^{\alpha}})/(1-x)$ and the compositions of α. Further, he considered perfect partitions of the number $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots - 1$, where $p_1, p_2, \ldots$ are primes, and found that the number of perfect partitions of this number is equal to the number of compositions of the multipartite number $(\alpha_1, \alpha_2, \ldots)$. The fact that the number of perfect partitions of n is the same as the number of ordered factorizations of $n+1$ was also shown.

A similar idea of representing numbers as a sum of given numbers was used in later days. It seems that the word "complete" first appeared in a problem suggested by Hoggatt and King in [3], which was solved in Brown's paper [2]. They called an arbitrary sequence $\{f_i\}_{i=1}^{\infty}$ of positive integers "complete" if every positive integer n could be represented in the form $n = \sum_{i=1}^{\infty} \alpha_i f_i$, where each α_i was either 0 or 1. Brown found a simple necessary and sufficient condition for the completeness of such sequences and showed that the Fibonacci numbers are characterized by certain properties involving completeness. His note also considered whether or not completeness was destroyed by the deletion of some terms.

Now, turning our attention to partitions of a positive integer, we apply completeness to partitions. Several properties, recurrence relations, and generating functions for complete partitions will be obtained.

2. COMPLETE PARTITIONS

We begin with a definition of partitions of a positive integer.

Definition 2.1: A partition of a positive integer n is a finite non-decreasing sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ such that $\sum_{i=1}^{k} \lambda_i = n$ and $\lambda_i > 0$ for all $i = 1, \ldots, k$. The λ_i are called the *parts of the partition* and k is called the *length of the partition*.

We sometimes write $\lambda = (1^{m_1} 2^{m_2} \ldots)$, which means there are exactly m_i parts equal to i in the partition λ. For example, there are five partitions of 4: $(1^4), (1^2\,2), (2^2), (1\ 3)$, and (4). We are now ready to define our main topic—complete partitions.

---

* This research has been supported in part by YSRF and by Non Directed Research Fund, Korea Research Foundation.

Definition 2.2: A *complete partition* of an integer n is a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of n, with $\lambda_1 = 1$, such that each integer i, $1 \le i \le n$, can be represented as a sum of elements of $\lambda_1, \ldots, \lambda_k$. In other words, each i can be expressed as $\sum_{j=1}^{k} \alpha_j \lambda_j$, where α_j is either 0 or 1.

Example 2.3: Among the five partitions of 4, (1^4) and $(1^2\, 2)$ are complete partitions of 4.

From Definition 2.2, the following is obvious.

Lemma 2.4: Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a complete partition of a positive integer. Then $(\lambda_1, \ldots, \lambda_i)$ is a complete partition of the number $\lambda_1 + \cdots + \lambda_i$ for $i = 1, \ldots, k$. □

Brown [2] found the following three facts on completeness of sequences of positive integers which are also true for partitions.

Proposition 2.5 (Brown [2]): Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a complete partition of a positive integer. Then, for $i = 1, \ldots, k-1$,

$$\lambda_{i+1} \le 1 + \sum_{j=1}^{i} \lambda_j .$$

Proof: Suppose not. Then there exists at least one $r \ge 2$ such that $\lambda_r > 1 + \sum_{i=1}^{r-1} \lambda_i$. Therefore, $\lambda_r > \lambda_r - 1 > \sum_{i=1}^{r-1} \lambda_i$. Thus, the integer $\lambda_r - 1$ cannot be represented as a sum of elements of $\lambda_1, \lambda_2, \ldots, \lambda_k$. □

The converse of Proposition 2.5 is also true, which we shall prove here in a manner different from that of Brown .

Theorem 2.6 (Brown [2]): Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of n with $\lambda_1 = 1$ such that

$$\lambda_{i+1} \le 1 + \sum_{j=1}^{i} \lambda_j, \text{ for } i = 1, \ldots, k-1.$$

Then λ is a complete partition of n.

Proof: Suppose not. Then there must be some numbers between 1 and n that cannot be expressed as a sum of elements of $\lambda_1, \ldots, \lambda_k$. Let m be the least such number. Then we have $\lambda_1 + \cdots + \lambda_i < m < \lambda_1 + \cdots + \lambda_i + \lambda_{i+1}$, for some $i \ge 1$. We claim that $m < \lambda_{i+1}$. From our choice of m, we know that $m \ne \lambda_{i+1}$. If $m > \lambda_{i+1}$, then $0 < m - \lambda_{i+1} < m < n$. So $m - \lambda_{i+1}$ must be represented in the form $\sum_{j=1}^{i} \alpha_j \lambda_j$, where $\alpha_i = 0$ or 1, which contradicts our choice of m. Therefore, $m < \lambda_{i+1}$, so $1 + \lambda_1 + \cdots + \lambda_i < a + m \le \lambda_{i+1}$, a contradiction. □

Corollary 2.7 (Brown [2]): Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a complete partition. Then $\lambda_i \le 2^{i-1}$ for each $i = 1, \ldots, k$.

Proof: Obviously it is true for $i = 1, \ldots, k$, since $\lambda_1 = 1 \le 2^0 = 1$. Assuming $\lambda_i \le 2^{i-1}$ for each $i = 1, \ldots, j$, we have $\lambda_{j+1} \le 1 + \sum_{\ell=1}^{j} \lambda_\ell \le 1 + 1 + 2 + 2^2 + \cdots + 2^{j-1} = 2^j$. □

Now let us characterize complete partitions by the length and the size of parts.

Proposition 2.8: Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a complete partition of a positive integer n. Then the minimum possible length k is $\lceil \log_2(n+1) \rceil$, where $\lceil x \rceil$ is the least integer $\ge x$.

Proof: By Corollary 2.7,

$$n=\sum_{i=1}^{k}\lambda_i\le\sum_{i=0}^{k-1}2^i=2^k-1.$$

Therefore, $n+1\le 2^k$, which gives $k\ge\lceil\log_2(n+1)\rceil$. □

Proposition 2.9 Let $\lambda=(\lambda_1,\dots,\lambda_k)$ be a complete partition of n. Then the largest possible part is $\lfloor\frac{n+1}{2}\rfloor$, where $\lfloor x\rfloor$ is the greatest integer $\le x$.

Proof: Straightforward from Theorem 2.6. □

3. RECURRENCE RELATIONS AND GENERATING FUNCTIONS

In this section we find some recurrence relations to count complete partitions of a positive integer n. Let $C_{\ell,k}(n)$ be the number of complete partitions of n with length ℓ and largest part k. Then, by Proposition 2.9 and Lemma 2.7, k and ℓ must satisfy

$$1\le k\le\lfloor\tfrac{n+1}{2}\rfloor \text{ and } \lceil\log_2(n+1)\rceil\le\ell\le n.$$

Obviously, $C_{\ell,k}(n)=1$ if $n=k+\ell-1$ and $C_{1,1}(1)=1$ in particular. Since k is the largest part, counting complete partitions of $n-k$ with length $\ell-1$ and largest parts from 1 to k gives the following recurrence relation.

Proposition 3.1: Let $C_{\ell,k}(n)$ be the number of complete partition of n with length ℓ and largest part k. Then

$$C_{\ell,k}(n)=\begin{cases}\sum_{i=1}^{k}C_{\ell-1,i}(n-k) & \text{if } 1\le k\le\lfloor\frac{n+1}{2}\rfloor \text{ and } \lceil\log_2(n+1)\rceil\le\ell\le n,\\ 0 & \text{otherwise,}\end{cases}$$

with the initial condition $C_{1,1}(1)=1$. □

Example 3.2: There are two complete partitions of 8 with length $\ell=4$ and largest part $k=3$: $(12^2 3)$ and $(1^2 3^2)$. This is obtained by the recurrence relation

$$\begin{aligned}C_{4,3}(8)&=\sum_{i=1}^{3}C_{3,i}(5)=C_{3,1}(5)+C_{3,2}(5)+C_{3,3}(5)\\&=C_{3,1}(5)+(C_{2,1}(3)+C_{2,2}(3))+C_{2,1}(2)\\&=0+(0+1)+1=2.\end{aligned}$$

Naturally, by adding $C_{\ell,k}(n)$ for all possible ℓ, we obtain the total number of complete partitions of n as follows.

Corollary 3.3: Let $C(n)$ be the number of complete partitions of n. Then

$$C(n)=\sum_{\ell=\lceil\log_2(n+1)\rceil}^{n}\ \sum_{k=1}^{\lfloor(n+1)/2\rfloor}C_{\ell,k}(n).\ \square$$

Now let us take a look at the size of parts to get another recurrence relation. Let $C_k(n)$ be the number of complete partitions of a positive integer n with largest part at most k. We take

$C_0(n)=0$ for all $n \geq 0$ and $C_k(0)=0$ for all $k \geq 1$, and $C_1(1)=1$. So $C_k(n)$ is always positive if $n, k \geq 1$; therefore, k ranges from 1 to $\lfloor \frac{n+1}{2} \rfloor$ by Proposition 2.9. From our definition of $C_k(n)$, for any $k > \lfloor \frac{n+1}{2} \rfloor$,

$$C_k(n) = C_{k-1}(n) = \cdots = C_{\lfloor \frac{n+1}{2} \rfloor}(n).$$

The set of complete partitions of n with largest part at most k can be partitioned into two subsets: one with largest part exactly k; the other with largest part at most $k-1$. It is not difficult to see that the number of complete partitions of n with largest part exactly k is equal to the number of complete partitions of $n-k$ with largest part at most k. Therefore, we have the following theorem.

Theorem 3.4: Let $C_k(n)$ be the number of complete partitions of a positive integer n with largest part at most k $(k \geq 1)$. Then

$$C_k(n) = \begin{cases} C_{k-1}(n) + C_k(n-k) & \text{if } 1 \leq k \leq \lfloor \frac{n+1}{2} \rfloor, \\ C_{\lfloor \frac{n+1}{2} \rfloor}(n) & \text{if } k > \lfloor \frac{n+1}{2} \rfloor, \end{cases}$$

with the initial conditions $C_0(n)=0$ for all $n \geq 0$, $C_k(0)=0$ for all k, and $C_1(1)=1$.

Note that $C_1(n)=1$ for all $n \geq 1$. Let us take some examples.

Example 3.5:

1. $C_2(5) = C_1(5) + C_2(3) = C_1(5) + (C_1(3) + C_2(1)) = C_1(5) + (C_1(3) + C_1(1)) = 1+1+1 = 3.$
2. $C_3(5) = C_2(5) + C_3(2) = C_2(5) + C_1(2) = 3+1 = 4.$

Corollary 3.6: If $k = \lfloor (n+1)/2 \rfloor$, then $C_k(n)$ is the number of all complete partitions of n.

Now we count complete partitions by the largest part. Let $D_k(n)$ be the number of complete partitions of a positive integer n with largest part exactly k. Then $D_k(n) = C_k(n) - C_{k-1}(n) = C_k(n-k)$ by the definition of $C_k(n)$ and Theorem 3.4. Obviously, $D_1(n)=1$ for all $n \geq 1$. Thus, for $k \geq 2$,

$$\begin{aligned} D_k(n) &= C_k(n-k) \\ &= C_k(n-k) - C_{k-1}(n-k) + C_{k-1}(n-k) \\ &= D_k(n-k) + D_{k-1}(n-1). \end{aligned}$$

Since each complete partition of n with largest part exactly k must have at least one k as a part, $D_k(n)=0$ if $1 \leq n \leq 2k-2$ and $D_k(n-k)=0$ if $2k-1 \leq n \leq 3k-2$. Thus, we obtain

Theorem 3.7: Let $D_k(n)$ be the number of complete partitions of a positive integer n with largest part exactly k. Then $D_1(n)=1$ for all $n \geq 1$ and for $k \geq 2$,

$$D_k(n) = \begin{cases} D_{k-1}(n-1) + D_k(n-k) & \text{if } n \geq 3k-1, \\ D_{k-1}(n-1) & \text{if } 2k-1 \leq n \leq 3k-2, \\ 0 & \text{if } 1 \leq n \leq 2k-2, \end{cases}$$

with the conditions $D_0(n)=0$ for all n and $D_k(0)=0$ for all k.

Example 3.8:

1. $D_2(4) = D_1(3) = 1;$
2. $D_2(5) = D_1(4) + D_2(3) = D_1(4) + D_1(3) = 1 + 1 = 2;$
3. $D_3(7) = D_2(6) = D_1(5) + D_2(4) = 1 + 1 = 2;$
4. $D_2(7) + D_3(5) = (D_1(6) + D_2(5)) + D_2(4) = (1+2) + 1 = 4.$

The following table shows the first few values of complete partitions of n with largest part at most k, and $C(n)$ is the total number of complete partitions of n.

$k \backslash n$	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1	1	1	1	1	1	1	1	1	1	1
2			1	1	2	2	3	3	4	4	5	5
3					1	2	2	4	5	6	8	10
4							2	2	4	5	8	10
5									2	4	5	8
6											4	5
$C(n)$	1	1	2	2	4	5	8	10	16	20	31	39

Now we find the generating function for the number $D_k(n)$.

Theorem 3.9: Let $f_k(q) = \sum_{n=0}^{\infty} D_k(n)q^n$ $(k \geq 2)$. Then we have

$$f_k(q) = \frac{q^{k+1}}{(1-q^k)(1-q^{k-1})\cdots(1-q)}$$
$$-\left(\frac{D_{k-1}(2k-3)q^{2k-2}}{1-q^k} + \frac{D_{k-2}(2k-5)q^{2k-3}}{(1-q^k)(1-q^{k-1})} + \cdots + \frac{q^{k+1}}{(1-q^k)(1-q^{k-1})\cdots(1-q^3)}\right),$$

with $f_1(q) = \frac{q}{1-q}$.

Proof: Since $D_1(n) = 1$ for all n, $f_1(q) = \frac{q}{1-q}$. Let $f_k(q) = \sum_{n=0}^{\infty} D_k(n)q^n$. Then

$$\begin{aligned}
f_k(q) &= \sum_{n=0}^{\infty} D_k(n)q^n = \sum_{n=2k-1}^{\infty} D_k(n)q^n \\
&= \sum_{n=2k-1}^{3k-2} D_{k-1}(n-1)q^n + \sum_{n=3k-1}^{\infty} [D_{k-1}(n-1) + D_k(n-k)]q^n \\
&= \sum_{n=2k-1}^{\infty} D_{k-1}(n-1)q^n + \sum_{n=3k-1}^{\infty} D_k(n-k)q^n
\end{aligned}$$

$$= q \sum_{n=2k-1}^{\infty} D_{k-1}(n-1)q^{n-1} + q^k \sum_{n=3k-1}^{\infty} D_k(n-k)q^{n-k}$$

$$= q[f_{k-1}(q) - D_{k-1}(2k-3)q^{2k-3}] + q^k f_k(q).$$

Thus, we have

$$f_k(q) = \frac{q}{1-q^k} f_{k-1}(q) - \frac{D_{k-1}(2k-3)q^{2k-2}}{1-q^k}.$$

An iteration gives

$$\begin{aligned} f_k(q) &= \frac{q}{1-q^k} f_{k-1}(q) - \frac{D_{k-1}(2k-3)q^{2k-2}}{1-q^k} \\ &= \frac{q}{1-q^k}\left[\frac{q}{1-q^{k-1}} f_{k-2}(q) - \frac{D_{k-2}(2k-5)q^{2k-4}}{1-q^{k-1}}\right] - \frac{D_{k-1}(2k-3)q^{2k-2}}{1-q^k} \\ &= \frac{q^2}{(1-q^k)(1-q^{k-1})} f_{k-2}(q) - \left[\frac{D_{k-1}(2k-3)q^{2k-2}}{1-q^k} + \frac{D_{k-2}(2k-5)}{(1-q^k)(1-q^{k-1})}\right]. \end{aligned}$$

By continuing iteration on $f_k(q)$, we obtain

$$\begin{aligned} f_k(q) &= \frac{q^{k-2}}{(1-q^k)(1-q^{k-1})\cdots(1-q^3)} f_2(q) \\ &\quad - \left[\frac{D_{k-1}(2k-3)q^{2k-2}}{1-q^k} + \frac{D_{k-2}(2k-5)q^{2k-3}}{(1-q^k)(1-q^{k-1})} + \cdots + \frac{D_2(3)q^{k+1}}{(1-q^k)(1-q^{k-1})\cdots(1-q^3}\right]. \end{aligned}$$

Since $D_2(3) = 1$, $f_1(q) = \frac{q}{1-q}$, and

$$f_2(q) = \frac{q^2}{(1-q^2)(1-q)} - \frac{q^2}{1-q^2} = \frac{q^3}{(1-q^2)(1-q)},$$

the theorem follows. □

Note that the numbers $D_{k-i}(2(k-i)-1)$, for $i = 1, 2, \ldots, k-2$, in $f_k(q)$ can be simplified to

$$D_{\lfloor \frac{k-i+1}{2} \rfloor}\left(\left\lfloor \frac{3(k-i)-1}{2} \right\rfloor\right)$$

by Theorem 3.7.

Example 3.10: The following are generating functions for $k = 3, 4$, and 5.

$$f_3(q) = \frac{q^4}{(1-q^3)(1-q^2)(1-q)} - \frac{q^4}{1-q^3},$$

$$f_4(q) = \frac{q^5}{(1-q^4)(1-q^3)(1-q^2)(1-q)} - \left[\frac{q^6}{1-q^4} + \frac{q^5}{(1-q^4)(1-q^3)}\right],$$

and

$$f_5(q) = \frac{q^6}{(1-q^5)(1-q^4)(1-q^3)(1-q^2)(1-q)} - \left[\frac{2q^8}{1-q^5} + \frac{q^7}{(1-q^5)(1-q^4)} + \frac{q^6}{(1-q^5)(1-q^4)(1-q^3)}\right].$$

By expanding the above, we get the following, which is expected from the above table.

$$f_3(q) = q^5 + 2q^6 + 2q^7 + 4q^8 + 5q^9 + 6q^{10} + 8q^{11} + 10q^{12} + \cdots,$$
$$f_4(q) = 2q^7 + 2q^8 + 4q^9 + 5q^{10} + 8q^{11} + 10q^{12} + \cdots,$$
$$f_5(q) = 2q^9 + 4q^{10} + 5q^{11} + 8q^{12} + \cdots.$$

ACKNOWLEDGMENT

I would like to thank Professor Ira Gessel of Brandeis University for his valuable comments and Professor James Lewis of the University of Rhode Island who brought this topic to my attention.

REFERENCES

1. G. E. Andrews. *The Theory of Partitions, Encyclopedia of Mathematics and Its Applications*. Vol. 2. Reading, Mass.: Addison-Wesley, 1976.
2. J. L. Brown. "Note on Complete Sequences of Integers." *Amer. Math. Monthly* **68** (1961): 557-60.
3. V. E. Hoggatt & C. King. "Problem E 1424." *Amer. Math. Monthly* **67** (1960):593.
4. P. A. MacMahon. *Combinatory Analysis*. Vols. I and II. Cambridge: Cambridge University Press, 1915, 1916 (rpt. Chelsea, 1960).

AMS Classification Number: 05A17

❖❖❖