GENERALIZATIONS OF SOME IDENTITIES INVOLVING GENERALIZED SECOND-ORDER INTEGER SEQUENCES

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In [4], using the method of Carlitz and Ferns [1], some identities involving generalized second-order integer sequences were given. The purpose of this paper is to obtain the more general results.

In the notation of Horadam [2], write $W_n = W_n(a,b; p,q)$ so that

$$W_n = pW_{n-1} - qW_{n-2},\ W_0 = a, W_1 = b,\ n \geq 2. \tag{1}$$

If α and β, assumed distinct, are the roots of $\lambda^2 - p\lambda + q = 0$, we have the Binet form (see [2])

$$W_n = A\alpha^n + B\beta^n, \tag{2}$$

where $A = \frac{b-a\beta}{\alpha-\beta}$ and $B = \frac{a\alpha-b}{\alpha-\beta}$.

Using this notation, define $U_n = W_n(0,1; p,q)$ and $V_n = W_n(2,p; p,q)$. The Binet forms for U_n and V_n are given by $U_n = (\alpha^n - \beta^n)/(\alpha-\beta)$ and $V_n = \alpha^n + \beta^n$, where $\{U_n\}$ and $\{V_n\}$ are the fundamental and primordial sequences, respectively. They have been studied extensively, particularly by Lucas [3].

Throughout this paper, the symbol $\binom{n}{i,j}$ is defined by $\binom{n}{i,j} = \frac{n!}{i!j!(n-i-j)!}$.

To extend the results of [4], we need the following lemma.

Lemma: Let $u = \alpha$ or β, then

$$-q^{m+1} + pq^m u + u^{2(m+1)} = V_m u^{m+2}. \tag{3}$$

Proof: Since α and β are roots of $\lambda^2 - p\lambda + q = 0$, we have $\alpha^2 = p\alpha - q$ and $\beta^2 = p\beta - q$. Hence,

$$\begin{aligned}-q^{m+1} + pq^m u + u^{2(m+1)} &= q^m(pu-q) + u^{2(m+1)} = q^m u^2 + u^{2(m+1)}\\ &= u^{m+2}(q^m u^{-m} + u^m) = (\alpha^m + \beta^m)u^{m+2} = V_m u^{m+2}.\end{aligned}$$

This completes the proof of the Lemma.

Theorem 1:

$$-q^{m+1}W_k + pq^m W_{k+1} + W_{k+2(m+1)} = V_m W_{k+m+2}. \tag{4}$$

Proof: By the Lemma, we have

$$-q^{m+1} + pq^m\alpha + \alpha^{2(m+1)} = V_m\alpha^{m+2} \quad \text{and} \quad -q^{m+1} + pq^m\beta + \beta^{2(m+1)} = V_m\beta^{m+2}.$$

Theorem 1 follows if we multiply both sides of the previous two identities by α^k and β^k, respectively, and use the Binet form (2).

Theorem 2:

$$W_{n+k} = (pq^m)^{-n} \sum_{i+j+s=n} \binom{n}{i,j} (-1)^j q^{(m+1)s} V_m^i W_{(m+2)i+2(m+1)j+k}. \tag{5}$$

$$W_{(m+2)n+k} = V_m^{-n} \sum_{i+j+s=n} \binom{n}{i,\, j} (-1)^s p^j q^{mj+(m+1)s} W_{2(m+1)i+j+k}. \tag{6}$$

$$W_{2(m+1)n+k} = \sum_{i+j+s=n} \binom{n}{i,\, j} (-1)^j p^j q^{(m+1)s+mj} V_m^i W_{(m+2)i+j+k}. \tag{7}$$

Proof: By using the Lemma and the multinomial theorem, we have

$$(pq^m)^n u^n = \sum_{i+j+s=n} \binom{n}{i,\, j} (-1)^j q^{(m+1)s} V_m^i u^{(m+2)i+2(m+1)j},$$

$$V_m^n u^{(m+2)n} = \sum_{i+j+s=n} \binom{n}{i,\, j} (-1)^s p^j q^{mj+(m+1)s} u^{2(m+1)i+j},$$

$$u^{2(m+1)n} = \sum_{i+j+s=n} \binom{n}{i,\, j} (-1)^j p^j q^{(m+1)s+mj} V_m^i u^{(m+2)i+j}.$$

If we multiply both sides in the preceding identities by u^k and use the Binet form (2), we obtain (5), (6), and (7), respectively. This completes the proof of Theorem 2.

Theorem 3:

$$p^n q^{mn} W_{n+k} - \sum_{j=0}^{n} \binom{n}{j} (-1)^j q^{(m+1)(n-j)} W_{2(m+1)j+k} \equiv 0 \pmod{V_m}. \tag{8}$$

$$W_{2(m+1)n+k} - (-1)^n q^{mn} W_{2n+k} \equiv 0 \pmod{V_m}. \tag{9}$$

Proof: From (5) and (7), by using the decomposition $\Sigma_{i+j+s=n} = \Sigma_{i+j+s=n,\, i=0} + \Sigma_{i+j+s=n,\, i\neq 0}$ and Theorem 2.1 of [4], we get Theorem 3.

Remark: When we take $m = 2, 4$, and 8, the results of this paper become those of [4].

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