ADVANCED PROBLEMS AND SOLUTIONS

Edited by Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-545 ***Proposed by Paul S. Bruckman, Highwood, IL***

Prove that for all odd primes p,

$$\textit{(a)}\ \sum_{k=1}^{p-1} L_k \cdot k^{-1} \equiv \frac{-2}{p}(L_p - 1) \pmod{p}; \qquad \textit{(b)}\ \sum_{k=1}^{p-1} F_k \cdot k^{-1} \equiv 0 \pmod{p}.$$

H-546 ***Proposed by R. André-Jeannin, Longwy, France***

Find the triangular Mersenne numbers (the sequence of Mersenne numbers is defined by $M_n = 2^n - 1$).

SOLUTIONS

A Prime Problem

H-528 ***Proposed by Paul S. Bruckman, Highwood, IL***
(Vol. 35, no. 2, May 1997)

Let $\Omega(n) = \sum_{p^e \| n} e$, given the prime decomposition of a natural number $n = \prod p^e$. Prove the following:

$$\sum_{d \mid n} (-1)^{\Omega(d)} F_{\Omega(n/d) - \Omega(d)} = 0; \tag{A}$$

$$\sum_{d \mid n} (-1)^{\Omega(d)} L_{\Omega(n/d) - \Omega(d)} = 2U_n, \text{ where } U_n = \prod_{p^e \| n} F_{e+1}. \tag{B}$$

Solution by H.-J. Seiffert, Berlin, Germany

Define the Fibonacci and Lucas polynomials by

$$F_0(x) = 0,\ F_1(x) = 1,\ F_{n+1}(x) = xF_n(x) + F_{n-1}(x),\ n \in Z,$$
$$L_0(x) = 2,\ L_1(x) = x,\ L_{n+1}(x) = xL_n(x) + L_{n-1}(x),\ n \in Z,$$

respectively. We shall prove that for all complex numbers x and all positive integers n,

$$\sum_{d \mid n} (-1)^{\Omega(d)} F_{\Omega(n/d) - \Omega(d)}(x) = 0; \tag{A'}$$

$$\sum_{d \mid n} (-1)^{\Omega(d)} L_{\Omega(n/d) - \Omega(d)}(x) = 2 \prod_{p^e \| n} F_{e+1}(x); \tag{B'}$$

$$\sum_{d\mid n} L_{2\Omega(n/d)-2\Omega(d)}(x) = 2x^{-\omega(n)} \prod_{p^e \| n} F_{2e+2}(x), \tag{C$'$}$$

where $\omega(n)$ denotes the number of distinct prime factors of n.

The desired identities (A) and (B) are obtained from (A$'$) and (B$'$), respectively, by taking $x=1$.

We need the following known equations [see A. F. Horadam & Bro. J. M. Mahon, "Pell and Pell-Lucas Polynomials, *The Fibonacci Quarterly* **23.1** (1985):7-20, equations (2.1), (3.23), and (3.25)],

$$L_j(x) = F_{j-1}(x) + F_{j+1}(x),\ j \in Z, \tag{1}$$

$$L_{j+k}(x) + (-1)^k L_{j-k}(x) = L_j(x) L_k(x),\ j, k \in Z, \tag{2}$$

and the easily verified relations,

$$L_{-j}(x) = (-1)^j L_j(x) \quad \text{and} \quad F_{-j}(x) = (-1)^{j-1} F_j(x),\ j \in Z.$$

Proposition: For all nonnegative integers m and e, we have

$$\sum_{j=0}^{m-1} (-1)^j L_{e-2j}(x) = F_{e+1}(x) - (-1)^{m-e} F_{2m-e-1}(x).$$

Proof: This is true for $m=0$ (empty sums have the value zero). Suppose that the equation holds for $m, m \in N_0$ (whole numbers). Then

$$\begin{aligned}\sum_{j=0}^{m} (-1)^j L_{e-2j}(x) &= \sum_{j=0}^{m-1} (-1)^j L_{e-2j}(x) + (-1)^m L_{e-2m}(x)\\ &= F_{e+1}(x) - (-1)^{m-e} F_{2m-e-1}(x) + (-1)^m L_{e-2m}(x)\\ &= F_{e+1}(x) - (-1)^{m+1-e}(L_{2m-e}(x) - F_{2m-e-1}(x))\\ &= F_{e+1}(x) - (-1)^{m+1-e} F_{2m-e+1}(x),\end{aligned}$$

where we have used (1). This completes the induction proof. Q.E.D.

Corollary: For all nonnegative integers e, we have

$$\sum_{j=0}^{e} (-1)^j L_{e-2j}(x) = 2F_{e+1}(x).$$

Proof: Take $m = e+1$ in the equation of the Proposition. Q.E.D.

Now we are able to prove the desired identities. We note that if d runs through all positive divisors of n, so does n/d. Hence, if $S(n)$ denotes the left side of (A$'$), then

$$S(n) = \sum_{d\mid n} (-1)^{\Omega(n/d)} F_{\Omega(d)-\Omega(n/d)}(x) = -\sum_{d\mid n} (-1)^{\Omega(d)} F_{\Omega(n/d)-\Omega(d)}(x) = -S(n),$$

or $S(n)=0$. This proves (A$'$).

The proof of (B$'$) is more interesting. Let $T(n)$ denote the left side of (B$'$). If $n = p^e$ is a prime power, then by the identity of the above Corollary,

$$T(n) = T(p^e) = \sum_{j=0}^{e} (-1)^j L_{e-2j}(x) = 2F_{e+1}(x).$$

Thus, (B′) holds for all prime powers n. The proof of (B′) is completed by showing that the function $f : N \to C$ defined by $f(n) = T(n)/2$, $n \in N$, is multiplicative. Let m and n be coprime natural numbers. If $c \mid m$ and $d \mid n$, then

$$\Omega\left(\frac{mn}{cd}\right) - \Omega(cd) = \Omega\left(\frac{m}{c}\right) - \Omega(c) + \Omega\left(\frac{n}{d}\right) - \Omega(d)$$

and

$$\Omega\left(\frac{m}{c}d\right) - \Omega\left(c\frac{n}{d}\right) = \Omega\left(\frac{m}{c}\right) - \Omega(c) + \Omega(d) - \Omega\left(\frac{n}{d}\right),$$

so that by (2),

$$(-1)^{\Omega(cd)} L_{\Omega(\frac{mn}{cd})-\Omega(cd)}(x) + (-1)^{\Omega(\frac{cn}{d})} L_{\Omega(\frac{m}{c}d)-\Omega(c\frac{n}{d})}(x) = (-1)^{\Omega(c)}(-1)^{\Omega(d)} L_{\Omega(\frac{m}{c})-\Omega(c)}(x) L_{\Omega(\frac{n}{d})-\Omega(d)}(x).$$

Summing over all positive divisors c of m and d of n, we obtain the claimed equation:

$$f(mn) = f(m) f(n).$$

This completes the proof of (B′).

The desired identity (C′) easily follows from (B′) when we replace x by $i(x^2+2)$, where $i = \sqrt{(-1)}$, and use the known relations

$$L_j(i(x^2+2)) = i^j L_{2j}(x)$$

and

$$F_j(i(x^2+2)) = i^{j-1} F_{2j}(x)/x, \ j \in Z.$$

Let us look at what we get from (B′) if we set $x = 2i$. Now, since $L_j(2i) = 2i^j$ and $F_j(2i) = ji^{j-1}$, $j \in Z$, (B′) gives, after some simplification,

$$\tau(n) = \sum_{d \mid n} 1 = \prod_{p^e \| n} (e+1),$$

where $\tau(n)$ denotes the number of positive divisors of n. This is a well-known identity from Analytic Number Theory.

Also solved by the proposer.

Triple Play

H-529 ***Proposed by Paul S. Bruckman, Highwood, IL***
(Vol. 35, no. 3, August 1997)

Let ρ denote the set of Pythagorean triples (a, b, c) such that $a^2 + b^2 = c^2$. Find all pairs of integers $m, n > 0$ such that $(a, b, c) = (F_m F_n, F_{m+1} F_{n+2}, F_{m+2} F_{n+1}) \in \rho$.

Solution by L. A. G. Dresel, Reading, England

Let $a = F_m F_n$, $b = F_{m+1} F_{n+2}$, $c = F_{m+2} F_{n+1}$. We shall prove that there is only one such Pythagorean triple with $m, n > 0$, namely $m = 3, n = 6$, giving $a = 16, b = 63, c = 65$. We use the identity

$5F_mF_n = L_{n+m} - (-1)^m L_{n-m}$, so that $5c = L_{n+m+3} + (-1)^{m+1} L_{n-m-1}$ and $5b = L_{n+m+3} - (-1)^{m+1} L_{n-m+1}$. Hence, $5(c+b) = 2L_{n+m+3} - (-1)^{m+1} L_{n-m}$ and $(c-b) = (-1)^{m+1} F_{n-m}$. Since F_t and F_{t+1} have no common factor, it follows that a, b, and c have no common factor, and the Pythagorean triples must take the form $2uv$, $u^2 - v^2$, $u^2 + v^2$, where $u > v > 0$, have no common factor; hence, c is odd, while just one of a and b is even. We now consider these two cases in turn.

Case A. Let $a = 2uv$, then b and c are odd, and we have $3 \mid m$ and $3 \mid n$, while $c - b = 2v^2$ gives $(-1)^{m+1} F_{n-m} = 2v^2$. Using a result proved by J. H. E. Cohn in [1], this implies that $|n-m| = 0$ or 6. We can reject $n = m$, since this gives $b = c$ and $a = 0$. Taking $|n-m| = 3$, we have $F_{\pm 3} = 2$ and $v = 1$, so that m must be odd. Furthermore, we have $5(c+b) = 10u^2$. Hence, if $n = m+3$, then $10u^2 = 2(L_{2(m+3)} - 2) = 10(F_{m+3})^2$ gives $u = F_{m+3}$; if $n = m-3$, then $10u^2 = 2(L_{2m} + 2) = 10(F_m)^2$ gives $u = F_m$, since m is odd. Also, $a = 2uv = 2u = 2F_{m+3}$ or $2F_{n+3}$. But we also have $a = F_mF_n$; therefore, the smaller factor must be $F_3 = 2$, and this must be F_m, since m is odd. Hence, $m = 3$ and $n = 6$ is the only solution when $|n-m| = 3$.

Next, take $|n-m| = 6$, so that $2v^2 = (-1)^{m+1} F_{n-m} = 8$. If $n - m = 6$, m must be odd, and we obtain $10u^2 = 2(L_{2m+9} - 9)$; then, since $3 \mid m$, $2m+9$ is an odd multiple of 3, and $4 \mid L_{2m+9}$. Therefore, $5u^2 \equiv u^2 \equiv -1 \pmod 4$, which shows that there are no solutions in this case.

Finally, if $n - m = -6$, m must be even, and we have $6 \mid m$ and $6 \mid n$, so that $F_6 \mid F_m$ and $F_6 \mid F_n$, making F_mF_n divisible by 64. But we have $2v^2 = 8$, giving $v = 2$, so that $a = 2uv = 4u$, where u is odd, since $(u, v) = 1$. Hence, it is not possible to satisfy $a = F_mF_n$ if $n - m = -6$.

Case B. Now, if $b = 2uv$, then $c - b = u^2 + v^2 - 2uv = (u-v)^2$, so that $(-1)^{m+1} F_{n-m} = (u-v)^2$. It was also proved by J. H. E. Cohn in [1] that this implies $|n-m| = 0, 1, 2$, or 12. But since a and c are odd, we must have both $3 \mid (m+1)$ and $3 \mid (n+2)$. This implies $3 \mid (n-m+1)$, which rules out $|n-m| = 0$ and 12, and we are left with $(-1)^{m+1} F_{n-m} = 1$. We then find that m must be odd, of the form $m = 6t - 1$ (with $t \geq 1$), while the corresponding n can be either $n = 6t+1$ or $n = 6t-2$. But $c - b = 1$, so that $a^2 = c^2 - b^2 = c + b$. Since $a = F_mF_n$, this gives

$$(L_{2m} + 2)(L_{2n} \pm 2) = 5\{2L_{n+m+3} - (-1)^{m+1} L_{n-m}\}.$$

Approximating by putting $L_r = \alpha^r$ and ignoring terms that are small compared to L_r, we obtain $\alpha^{2(m+n)} = 10\alpha^{n+m+3}$ approximately, and since $\alpha^5 > 11$, our equation gives $\alpha^{m+n} < 11\alpha^3 < \alpha^8$. But the smallest pair of values for m and n is given above as $m = 5$ and $n = 4$, giving $m + n = 9$. This gives a contradiction, and proves that there are no acceptable solutions in Case B.

Reference

1. J.H.E. Cohn. "Square Fibonacci Numbers, etc." *The Fibonacci Quarterly* **2.2** (1964):109-13.

Also solved by H.-J. Seiffert, I. Strazdins, and the proposer.

Some Period

H-530 ***Proposed by Andrej Dujella, University of Zagreb, Croatia***
(Vol. 35, no. 3, August 1997)

Let $k(n)$ be the period of a sequence of Fibonacci numbers $\{F_i\}$ modulo n. Prove that $k(n) \leq 6n$ for any positive integer n. Find all positive integers n such that $k(n) = 6n$.

Solution by Paul S. Bruckman, Highwood, IL

For the first part of the problem, it suffices to prove the following lemma.

Lemma 1: For all odd n, $k(n) \le 4n$.

Of course, $k(1) = 1$, hence the result is trivially true for $n = 1$. If $n > 1$ is odd, let K_e denote $k(2^e n)$, $N_e = 2^e n$, $k = k(n)$, and $R_e = K_e / N_e$. Assuming the result of Lemma 1, $K_1 = \text{LCM}(3, k) \le 3k$, hence $R_1 \le 3k/2n \le 6$, Next, $K_2 = \text{LCM}(6, k) \le 6k$, hence $R_2 \le 6k/4n \le 6$. Next, $K_3 = \text{LCM}(6, k) \le 6k$, hence $R_3 \le 6k/8n \le 3$. Finally, if $e \ge 4$, $K_e = \text{LCM}(3 \cdot 2^{e-1}, k) \le 3k \cdot 2^{e-1}$, hence $R_e \le 3k/2n \le 6$. Thus, the result of Lemma 1 implies that $k(n) \le 6n$ for all $n > 1$; it therefore suffices to prove Lemma 1.

Proof of Lemma 1: We first assume that $\gcd(n, 10) = 1$. The following results are well known for all primes $p \ne 2, 5$: $k(n)$ is even for all $n > 2$; $k(p) \mid (p-1)$ if $(5/p) = 1$, $k(p) \mid (2p+2)$ if $(5/p) = -1$. Also, $k(p^e) = p^{e-t} k(p)$ for some t with $1 \le t \le e$. Therefore, if $(5/p) = 1$, $k(p^e) = 2p^{e-t}(p-1)/2a$ for some integer a, while if $(5/p) = -1$, $k(p^e) = 4p^{e-t}(p+1)/2a$ for some integer a. If $n = \prod p^e$, $k(n) = \text{LCM}\{k(p^e)\}$. We then see that $k(n) \le 4\prod_{p^e \| n} p^{e-1}(p+1)/2$. Then $k(n)/n \le 4\prod_{p|n}(p+1)/2p < 4$, since $(p+1)/2p < 1$ for all p.

On the other hand, if we assume that $n = 5^e$, then $Z(n) = n$ and $k(n) = 4n$. If $n = 5^e m$, where $\gcd(m, 10) = 1$, then $k(n) = \text{LCM}(k(5^e), k(m)) = \text{LCM}(4 \cdot 5^e, k(m)) < 4n$. This proves Lemma 1. In conjunction with our earlier discussion, it follows that $k(n) \le 6n$ for all n.

From Lemma 1 and the earlier discussion, it is seen that the upper bound of $6n$ is possibly reached only if $n = 2^a 5^b$ for some integers a and b. Note that

$$k(2 \cdot 5^b) = \text{LCM}(3, 4 \cdot 5^b) = 12 \cdot 5^b = 6n.$$

Next,

$$k(4 \cdot 5^b) = k(8 \cdot 5^b) = \text{LCM}(6, 4 \cdot 5^b) = 12 \cdot 5^b = 3n \text{ or } 3n/2 < 6n.$$

Finally, if $a \ge 4$,

$$k(n) = \text{LCM}(3 \cdot 2^{a-1}, 4 \cdot 5^b) = 3 \cdot 2^{a-1} \cdot 5^b = 3n/2 < 6n.$$

Thus, $k(n) = 6n$ if and only if $n = 2 \cdot 5^b$, $b = 1, 2, \ldots$.

Also solved by D. Bloom, L. Dresel, and the proposer.

A Rational Decision

H-531 ***Proposed by Paul S. Bruckman, Highwood, IL***
(Vol. 35, no. 3, August 1997)

Consider the sum $S = \sum_{n=1}^{\infty} t(n)/n^2$, where $t(1) = 1$ and $t(n) = \prod_{p|n}(1-p^{-2})^{-1}$, $n > 1$, the product taken over all prime p dividing n. Evaluate S and show that it is rational.

Solution by H.-J. Seiffert, Berlin, Germany

We need the following results.

Theorem 1: If $f: N \to C$ is a multiplicative function such that $\sum_{n=1}^{\infty} f(n)/n^s$ converges absolutely for $\sigma = \text{Re}(s) > \sigma_0$, then

$$\sum_{n=1}^{\infty} f(n)/n^s = \prod_p \left(1+\sum_{j=1}^{\infty} f(p^j)/p^{js}\right) \text{ for } \sigma > \sigma_0,$$

where the product is over all primes p.

Proof: See ([1], pp. 230-31).

Theorem 2: For $\sigma > 1$, we have

$$\prod_p (1-p^{-s}) = 1/\zeta(s) \quad \text{and} \quad \prod_p (1+p^{-s}) = \zeta(s)/\zeta(2s),$$

where ζ denotes the Riemann Zeta function.

Proof: See ([1], p. 231).

Let $S_k = \sum_{n=1}^{\infty} t_k(n)/n^k$, $k \in C$, $\mathrm{Re}(k) > 1$, where $t_k(1) = 1$ and $t_k(n) = \prod_{p|n}(1-p^{-k})^{-1}$ for $n > 1$. Clearly, t_k is a multiplicative function. Since $t_k(p^j) = (1-p^{-k})^{-1}$ for all $j \in N$ and all primes p, we have

$$\sum_{j=1}^{\infty} t_k(p^j)/p^{jk} = p^{-k}(1-p^{-k})^{-2} \text{ for all primes } p,$$

where we have used the closed form expression for infinite geometric sums. Using

$$1+p^{-k}(1-p^{-k})^{-2} = (1-p^{-k})^{-1}(1-p^{-2k})^{-1}(1+p^{-3k}),$$

it follows from Theorems 1 and 2 that

$$S_k = \zeta(k)\zeta(2k)\zeta(3k)/\zeta(6k), \; k \in C, \; \mathrm{Re}(k) > 1. \tag{1}$$

Since ([1], p. 266)

$$\zeta(2j) = (-1)^{j+1}\frac{(2\pi)^{2j}}{2(2j)!}B_{2j}, \; j \in N,$$

where the B's are the Bernoulli numbers defined by ([1], p. 265, or [2], p. 9)

$$B = 1 \quad \text{and} \quad B_n = \sum_{r=0}^{n}\binom{n}{r}B_r, \; n \in N, \; n \geq 2,$$

from (1) we obtain

$$S_{2j} = \frac{(12j)!}{4(2j)!(4j)!(6j)!}\,\frac{B_{2j}B_{4j}B_{6j}}{B_{12j}}, \; j \in N, \tag{2}$$

showing that S_{2j}, $j \in N$, is a rational number. Using the values ([2], p. 10) $B_2 = \frac{1}{6}$, $B_4 = \frac{-1}{30}$, $B_6 = \frac{1}{42}$, and $B_{12} = \frac{-691}{2730}$, from (2) it is easily calculated that $S = S_2 = \frac{5005}{2764}$. This solves the present proposal.

References

1. T.M. Apostol. *Introduction to Analytic Number Theory*. New York: Springer-Verlag, 1976.
2. H. Rademacher. *Topics in Analytic Number Theory*. New York: Springer-Verlag, 1973.

Also solved by K. Lau, and the proposer.

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