HARMONIC SEEDS

G. L. Cohen and R. M. Sorli

School of Mathematical Sciences, University of Technology, Sydney
PO Box 123, Broadway, NSW 2007, Australia

(*Submitted December 1996*)

1. INTRODUCTION

Harmonic numbers were introduced by Ore [6] in 1948, though not under that name. A natural number n is harmonic if the harmonic mean of its positive divisors is an integer. Equivalently, n is harmonic if $H(n)$ is an integer, where

$$H(n) = \frac{n\tau(n)}{\sigma(n)},$$

and $\tau(n)$, $\sigma(n)$, respectively, are the number of and sum of the positive divisors of n.

Ore listed all harmonic numbers up to 10^5, and this list was extended by Garcia [3] to 10^7 and by Cohen [2] to $2 \cdot 10^9$. The second author of this paper has continued the list up to 10^{10}. In all of these cases, straightforward direct searches were used. No odd harmonic numbers have been found, giving the main interest to the topic since all perfect numbers are easily shown to be harmonic. If it could be proved that there are no odd harmonic numbers, then it would follow that there are no odd perfect numbers.

A number might be labeled also as arithmetic or geometric if the arithmetic mean, or geometric mean, respectively, of its positive divisors were an integer. Most harmonic numbers, but not all, appear to be also arithmetic (see Cohen [2]). It is easy to see that the set of geometric numbers is in fact simply the set of perfect squares, and it is of interest, according to Guy [4], that no harmonic numbers are known that are also geometric.

Although it is impractical to extend the direct search for harmonic numbers, we shall show, through the introduction of harmonic seeds, that no harmonic number less than 10^{12} is powerful. (We say that n is powerful if $p \mid n$ implies $p^2 \mid n$, where p is prime.) In particular, then, no harmonic number less than 10^{12} is also geometric. We have also used harmonic seeds to show that there are no odd harmonic numbers less than 10^{12}.

To define harmonic seeds, we first recall that d is a unitary divisor of n (and n is a unitary multiple of d) if $d \mid n$ and $\gcd(d, n/d) = 1$; we call the unitary divisor d proper if $d > 1$. Then:

Definition: A harmonic number (other than 1) is a harmonic seed if it does not have a smaller proper unitary divisor which is harmonic (and 1 is deemed to be the harmonic seed only of 1).

Then any harmonic number is either itself a harmonic seed or is a unitary multiple of a harmonic seed. For example, $n = 2^3 3^3 5^2 31$ is harmonic (with $H(n) = 27$); the proper unitary divisors of n are the various products of $2^3, 3^3, 5^2$, and 31. Since $2^3 5^2 31$ is harmonic and does not itself have a proper unitary harmonic divisor, it is a harmonic seed of n. (We are unable to prove that a harmonic number's harmonic seed is always unique, but conjecture that this is so.)

It is not as difficult to generate harmonic seeds only, and our two results on harmonic numbers less than 10^{12}, that (except for 1) none are powerful and none are odd, will clearly follow when the corresponding properties are seen to be true of the harmonic seeds less than 10^{12}.

2. COMPUTATION AND USE OF HARMONIC SEEDS

We shall need the following lemmas. Always, p and q will denote primes. We write $p^a \| n$ to mean $p^a \mid n$ and $p^{a+1} \nmid n$, and we then call p^a a component of n.

Lemma 1: Besides 1, the only squarefree harmonic number is 6 (Ore [6]).

Lemma 2: There are no harmonic numbers of the form p^a (Ore [6]). The only harmonic numbers of the form $p^a q^b$, $p \neq q$, are perfect numbers (Callan [1], Pomerance [7]).

Lemma 3: If n is an odd harmonic number and $p^a \| n$, then $p^a \equiv 1 \pmod 4$ (Garcia [3], Mills [5]).

Lemma 4: If n is an odd harmonic number greater than 1, then n has a component exceeding 10^7 (Mills [5]).

We first illustrate the algorithm for determining all harmonic seeds less than 10^{12}.

By Lemma 2, even perfect numbers are harmonic seeds and all other harmonic seeds, besides 1, have at least three distinct prime factors. Then, in seeking harmonic seeds n with $2^a \| n$, we must have $a \leq 35$, since $2^{36}3 \cdot 5 > 10^{12}$.

We build even harmonic seeds n, based on specific components 2^a, $1 \leq a \leq 35$, by calculating $H(n)$ simultaneously with n until $H(n)$ is an integer, using the denominators in the values of $H(n)$ to determine further prime factors of n. This uses the fact that H, like σ and τ, is multiplicative. For example, taking $a = 13$,

$$H(2^{13}) = \frac{2^{13}\tau(2^{13})}{\sigma(2^{13})} = \frac{2^{14}7}{3 \cdot 43 \cdot 127}.$$

Choosing the largest prime in the denominator, either $127^b \| n$ for $1 \leq b \leq 3$ (since $2^{13}3 \cdot 127^4 > 10^{12}$), or $p^{126} \mid n$ for some prime p so that $127 \mid \tau(n)$. In the latter case, clearly $n > 10^{12}$. With $b = 1$, we have

$$H(2^{13}127) = \frac{2^8 7}{3 \cdot 43},$$

so that $43^c \| n$ for $1 \leq c \leq 3$ (since $2^{13}43^4 127 > 10^{12}$) or $p^{42} \mid n$. In similar fashion, we then take, in particular,

$$H(2^{13}127 \cdot 43) = \frac{2^7 7}{3 \cdot 11}, \quad H(2^{13}127 \cdot 43 \cdot 11) = \frac{2^6 7}{3^2}.$$

At this stage we must have either $3^d \| n$ for $1 \leq d \leq 6$, or $p^2 \mid n$ for *two* primes p, or $p^8 \mid n$ for some prime p. All possibilities must be considered. We find

$$H(2^{13}127 \cdot 43 \cdot 11 \cdot 3^3) = \frac{2^5 3 \cdot 7}{5}, \quad H(2^{13}127 \cdot 43 \cdot 11 \cdot 3^3 5) = 2^5 7,$$

and so $2^{13}3^3 5 \cdot 11 \cdot 43 \cdot 127$ must be a harmonic seed.

Odd harmonic seeds up to 10^{12} were sought in the same way. Each odd prime was considered in turn as the smallest possible prime factor of an odd harmonic number. By Lemma 4, only the primes less than 317 needed to be considered since $317 \cdot 331 \cdot 10^7 > 10^{12}$. Lemmas 1 and 3 were also taken into account.

The list of all harmonic seeds less than 10^{12} is given in Table 1. Inspection of this list allows us to conclude the following.

Theorem 1: There are no powerful harmonic numbers less than 10^{12}.

Theorem 2: There are no odd harmonic numbers less than 10^{12}.

We had hoped originally that we would be able to generate easily all harmonic numbers, up to some bound, with a given harmonic seed. This turns out to be the case for those harmonic numbers which are squarefree multiples of their harmonic seed. In fact, we have the following result.

Theorem 3: Suppose n and $nq_1 \dots q_t$ are harmonic numbers, where $q_1 < \cdots < q_t$ are primes not dividing n. Then nq_1 is harmonic, except when $t \geq 2$ and $q_1 q_2 = 6$, in which case $nq_1 q_2$ is harmonic.

Proof: We may assume $t \geq 2$. Suppose first that $q_1 \geq 3$. Since $nq_1 \dots q_t$ is harmonic and H is multiplicative,

$$H(nq_1 \dots q_t) = H(n)H(q_1) \dots H(q_t) = H(n)\frac{2q_1}{q_1+1} \cdots \frac{2q_t}{q_t+1} = h,$$

say, where h is an integer. Then

$$H(n)q_1 \dots q_t = h\frac{q_1+1}{2} \cdots \frac{q_t+1}{2}.$$

Since $(q_1+1)/2 < \cdots < (q_t+1)/2 < q_t$, we have $q_t \mid h$, and then

$$H(nq_1 \dots q_{t-1}) = H(n)\frac{2q_1}{q_1+1} \cdots \frac{2q_{t-1}}{q_{t-1}+1} = \frac{h}{q_t}\frac{q_t+1}{2},$$

an integer. Applying the same argument to the harmonic number $nq_1 \dots q_{t-1}$, and repeating it as necessary, leads to our result in this case. In the less interesting case when $q_1 = 2$ (since then n must be an odd harmonic number), we again find that nq_1 is harmonic except perhaps if $q_2 = 3$, in which case $nq_1 q_2$ is harmonic. These details are omitted.

The point of Theorem 3 is that harmonic squarefree multiples of harmonic seeds may be built up a prime at a time. Furthermore, when n and nq_1 are harmonic numbers, with $q_1 > 2, q_1 \nmid n$, we have

$$H(nq_1)\frac{q_1+1}{2} = H(n)q_1,$$

so that $(q_1+1)/2 \mid H(n)$. Thus, $q_1 \leq 2H(n) - 1$, implying a relatively short search for all possible q_1, and then for q_2, and so on.

There does not seem to be a corresponding result for non-squarefree multiples of harmonic seeds. For example, $2^6 3^2 5 \cdot 13^3 17 \cdot 127$ is harmonic, but no unitary divisors of this number other than its seed $2^6 127$ and 1 are harmonic.

As an example of the application of Theorem 3, in Table 2 we give a list of all harmonic numbers n that are squarefree multiples of the seed 2457000. It is not difficult to see that the list is complete, and in fact it seems clear that there are only finitely many harmonic squarefree multiples of any harmonic number, all obtainable by the algorithm described above. However, a proof of this statement appears to be difficult.

TABLE 1

Harmonic seeds n less than 10^{12}			
n	$H(n)$	n	$H(n)$
1	1	$2876211000 = 2^3 3^2 5^3 13^2 31 \cdot 61$	150
$6 = 2 \cdot 3$	2	$8410907232 = 2^5 3^2 7^2 13 \cdot 19^2 127$	171
$28 = 2^2 7$	3	$8589869056 = 2^{16} 131071$	17
$270 = 2 \cdot 3^3 5$	6	$8628633000 = 2^3 3^3 5^3 13^2 31 \cdot 61$	195
$496 = 2^4 31$	5	$8698459616 = 2^5 7^2 11^2 19^2 127$	121
$672 = 2^5 3 \cdot 7$	8	$10200236032 = 2^{14} 7 \cdot 19 \cdot 31 \cdot 151$	96
$1638 = 2 \cdot 3^2 7 \cdot 13$	9	$14182439040 = 2^7 3^4 5 \cdot 7 \cdot 11^2 17 \cdot 19$	384
$6200 = 2^3 5^2 31$	10	$19017782784 = 2^9 3^2 7^2 11 \cdot 13 \cdot 19 \cdot 31$	336
$8128 = 2^6 127$	7	$19209881600 = 2^{11} 5^2 7^2 13 \cdot 19 \cdot 31$	256
$18620 = 2^2 5 \cdot 7^2 19$	14	$35032757760 = 2^9 3^2 5 \cdot 7^3 11 \cdot 13 \cdot 31$	392
$30240 = 2^5 3^3 5 \cdot 7$	24	$43861478400 = 2^{10} 3^3 5^2 23 \cdot 31 \cdot 89$	264
$32760 = 2^3 3^2 5 \cdot 7 \cdot 13$	24	$57575890944 = 2^{13} 3^2 11 \cdot 13 \cdot 43 \cdot 127$	192
$173600 = 2^5 5^2 7 \cdot 31$	25	$57648181500 = 2^2 3^2 5^3 7^3 13^3 17$	273
$1089270 = 2 \cdot 3^2 5 \cdot 7^2 13 \cdot 19$	42	$66433720320 = 2^{13} 3^3 5 \cdot 11 \cdot 43 \cdot 127$	224
$2229500 = 2^2 5^3 7^3 13$	35	$71271827200 = 2^8 5^2 7 \cdot 19 \cdot 31 \cdot 37 \cdot 73$	270
$2457000 = 2^3 3^3 5^3 7 \cdot 13$	60	$73924348400 = 2^4 5^2 7 \cdot 31^2 83 \cdot 331$	125
$4713984 = 2^9 3^3 11 \cdot 31$	48	$77924700000 = 2^5 3^3 5^5 7^2 19 \cdot 31$	375
$6051500 = 2^2 5^3 7^2 13 \cdot 19$	50	$81417705600 = 2^7 3 \cdot 5^2 7 \cdot 11^2 17 \cdot 19 \cdot 31$	484
$8506400 = 2^5 5^2 7^3 31$	49	$84418425000 = 2^3 3^2 5^5 7^2 13 \cdot 19 \cdot 31$	375
$17428320 = 2^5 3^2 5 \cdot 7^2 13 \cdot 19$	96	$109585986048 = 2^9 3^7 7 \cdot 11 \cdot 31 \cdot 41$	324
$23088800 = 2^5 5^2 7^2 19 \cdot 31$	70	$110886522600 = 2^3 3 \cdot 5^2 7 \cdot 31^2 83 \cdot 331$	155
$29410290 = 2 \cdot 3^5 5 \cdot 7^2 13 \cdot 19$	81	$124406100000 = 2^5 3^2 5^5 7^3 13 \cdot 31$	375
$33550336 = 2^{12} 8191$	13	$137438691328 = 2^{18} 524287$	19
$45532800 = 2^7 3^3 5^2 17 \cdot 31$	96	$156473635500 = 2^2 3^2 5^3 7^2 13^3 17 \cdot 19$	390
$52141320 = 2^3 3^4 5 \cdot 7 \cdot 11^2 19$	108	$183694492800 = 2^7 3^2 5^2 7^2 13 \cdot 17 \cdot 19 \cdot 31$	672
$81695250 = 2 \cdot 3^3 5^3 7^2 13 \cdot 19$	105	$206166804480 = 2^{11} 3^2 5 \cdot 7 \cdot 13^2 31 \cdot 61$	384
$115048440 = 2^3 3^2 5 \cdot 13^2 31 \cdot 61$	78	$221908282624 = 2^8 7 \cdot 19^2 37 \cdot 73 \cdot 127$	171
$142990848 = 2^9 3^2 7 \cdot 11 \cdot 13 \cdot 31$	120	$271309925250 = 2 \cdot 3^7 5^3 7^2 13 \cdot 19 \cdot 41$	405
$255428096 = 2^9 7 \cdot 11^2 19 \cdot 31$	88	$428440390560 = 2^5 3^2 5 \cdot 7^2 13^2 19 \cdot 31 \cdot 61$	546
$326781000 = 2^3 3^3 5^3 7^2 13 \cdot 19$	168	$443622427776 = 2^7 3^4 11^3 17 \cdot 31 \cdot 61$	352
$459818240 = 2^8 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73$	96	$469420906500 = 2^2 3^3 5^3 7^2 13^3 17 \cdot 19$	507
$481572000 = 2^5 3^3 5^3 7^3 13$	168	$513480135168 = 2^9 3^5 7^2 11 \cdot 13 \cdot 19 \cdot 31$	648
$644271264 = 2^5 3^2 7 \cdot 13^2 31 \cdot 61$	117	$677701763200 = 2^7 5^2 7 \cdot 11 \cdot 17^2 31 \cdot 307$	340
$1307124000 = 2^5 3^3 5^3 7^2 13 \cdot 19$	240	$830350521000 = 2^3 3^4 5^3 7^3 11^2 13 \cdot 19$	756
$1381161600 = 2^7 3^2 5^2 7 \cdot 13 \cdot 17 \cdot 31$	240	$945884459520 = 2^9 3^5 5 \cdot 7^3 11 \cdot 13 \cdot 31$	756
$1630964808 = 2^3 3^4 11^3 31 \cdot 61$	99	$997978703400 = 2^3 3^3 5^2 7 \cdot 31^2 83 \cdot 331$	279
$1867650048 = 2^{10} 3^4 11 \cdot 23 \cdot 89$	128		

Of the harmonic seeds in Table 1, the most prolific in producing harmonic squarefree multiples is 513480135168, with 216 such multiples. The largest is the 32-digit number

$$\begin{aligned} N_1 &= 29388663214285910932405215567360 \\ &= 2^9 3^5 5 \cdot 7^2 11 \cdot 13 \cdot 19 \cdot 23 \cdot 31 \cdot 137 \cdot 821 \cdot 8209 \cdot 16417 \cdot 32833, \end{aligned}$$

with $H(N_1) = 65666$. Much larger harmonic numbers were given by Garcia [3] and the algorithm above can be applied to give a great many harmonic squarefree multiples of those which are of truly gigantic size. Furthermore, most known multiperfect numbers (those n for which $\sigma(n) = kn$, for some integer $k \geq 2$) are also harmonic, for these are known only for $k \leq 10$ and nearly always it is the case that $k \mid \tau(n)$. For example, the largest known 4-perfect number (i.e., having $k = 4$) is

$$N_2 = 2^{37}3^{10}7 \cdot 11 \cdot 23 \cdot 83 \cdot 107 \cdot 331 \cdot 3851 \cdot 43691 \cdot 174763 \cdot 524287;$$

this has 169 harmonic squarefree multiples, the largest of which is

$$N_3 = N_2 \cdot 31 \cdot 61 \cdot 487 \cdot 3343 \cdot 3256081 \cdot 6512161 \cdot 13024321 \approx 5.53 \cdot 10^{73}$$

with $H(N_3) = 13024321$.

TABLE 2

All squarefree harmonic multiples n of 2457000	
n	$H(n)$
$27027000 = 2^3 3^3 5^3 7 \cdot 11 \cdot 13$	110
$513513000 = 2^3 3^3 5^3 7 \cdot 11 \cdot 13 \cdot 19$	209
$18999981000 = 2^3 3^3 5^3 7 \cdot 11 \cdot 13 \cdot 19 \cdot 37$	407
$1386998613000 = 2^3 3^3 5^3 7 \cdot 11 \cdot 13 \cdot 19 \cdot 37 \cdot 73$	803
$1162161000 = 2^3 3^3 5^3 7 \cdot 11 \cdot 13 \cdot 43$	215
$2945943000 = 2^3 3^3 5^3 7 \cdot 11 \cdot 13 \cdot 109$	218
$2457000 = 2^3 3^3 5^3 7 \cdot 13$	60
$46683000 = 2^3 3^3 5^3 7 \cdot 13 \cdot 19$	114
$1727271000 = 2^3 3^3 5^3 7 \cdot 13 \cdot 19 \cdot 37$	222
$126090783000 = 2^3 3^3 5^3 7 \cdot 13 \cdot 19 \cdot 37 \cdot 73$	438
$765181053000 = 2^3 3^3 5^3 7 \cdot 13 \cdot 19 \cdot 37 \cdot 443$	443
$5275179000 = 2^3 3^3 5^3 7 \cdot 13 \cdot 19 \cdot 113$	226
$10597041000 = 2^3 3^3 5^3 7 \cdot 13 \cdot 19 \cdot 227$	227
$56511000 = 2^3 3^3 5^3 7 \cdot 13 \cdot 23$	115
$12941019000 = 2^3 3^3 5^3 7 \cdot 13 \cdot 23 \cdot 229$	229
$5914045683000 = 2^3 3^3 5^3 7 \cdot 13 \cdot 23 \cdot 229 \cdot 457$	457
$71253000 = 2^3 3^3 5^3 7 \cdot 13 \cdot 29$	116
$144963000 = 2^3 3^3 5^3 7 \cdot 13 \cdot 59$	118

REFERENCES

1. D. Callan. Solution to Problem 6616. *Amer. Math. Monthly* **99** (1992):783-89.
2. G. L. Cohen. "Numbers Whose Positive Divisors Have Small Integral Harmonic Mean." *Math. Comp.* **66** (1997):883-91.
3. M. Garcia. "On Numbers with Integral Harmonic Mean." *Amer. Math. Monthly* **61** (1954): 89-96.
4. R. K. Guy. *Unsolved Problems in Number Theory*. 2nd ed. New York: Springer-Verlag, 1994.
5. W. H. Mills. "On a Conjecture of Ore." In *Proceedings of the 1972 Number Theory Conference*, pp. 142-46. University of Colorado, Boulder, Colorado, 1972.
6. O. Ore. "On the Average of the Divisors of a Number." *Amer. Math. Monthly* **55** (1948): 615-19.
7. C. Pomerance. "On a Problem of Ore: Harmonic Numbers." Unpublished manuscript, 1973.

AMS Classification Numbers: 11A25, 11Y70