LACUNARY RECURRENCES FOR SUMS OF POWERS OF INTEGERS

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1. INTRODUCTION

Let k and n be nonnegative integers with $n > 0$ and let

$$S_n(k) = 1^k + 2^k + \cdots + n^k. \tag{1.1}$$

Thus, $S_n(0) = n$, $S_n(1) = n(n+1)/2$, $S_n(2) = n(n+1)(2n+1)/6$, and so forth. A well-known recurrence is

$$\sum_{j=0}^{k-1} \binom{k}{j} S_n(j) = (1+n)^k - 1 \quad (k \geq 1).$$

It is also known (and easy to prove) that

$$2\sum_{j=0}^{k-1} \binom{2k}{2j} S_n(2j) = (1+n)^{2k} - n^{2k} - 1 \quad (k \geq 1), \tag{1.2}$$

$$2\sum_{j=0}^{k-1} \binom{2k+1}{2j+1} S_n(2j+1) = (1+n)^{2k+1} - n^{2k+1} - 1 \quad (k \geq 1) \tag{1.3}$$

(see, e.g., [8, p. 160]). Howard [4] proved the following formula. For $r = 0, 1, \ldots, 5$ and $n > 0$, $k \geq 1$:

$$6\sum_{j=0}^{k-1} \binom{6k+r-3}{6j+r} S_n(6j+r) = \sum_{s=1}^{6k+r-5} \binom{6k+r-3}{s} w_{r-s} n^s, \tag{1.4}$$

where $w_j = w_{6+j}$ for $j = 0, \pm 1, \pm 2, \ldots$, and the values of w_j for $j = 0, 1, \ldots, 5$ are given by 3, 2, 0, -1, 0, and 2, respectively.

These formulas suggest there may be other simple recurrences involving only $S_n(mj+r)$, where m, n, and r are fixed and $0 \leq r \leq m-1$. We call such formulas "lacunary," meaning they have lacunae, or gaps. That is, the value of $S_n(mk+r)$ does not depend on *all* the previous $S_n(j)$ $(0 \leq j \leq mk+r)$, but only on the terms $S_n(mj+r)$ $(0 \leq j \leq k)$.

In the present paper the main result is Theorem 3.1, which is a general lacunary recurrence for the sums $S_n(mj+r)$. After proving Theorem 3.1 in Section 3, we illustrate it by proving the following theorem for $m = 4$.

Theorem 1.1: For $k \geq 1$ and $r = 0, 1, 2, 3$,

$$4\sum_{j=0}^{k-1} \binom{4k+r}{4j+r} [(-4)^{k-j} - 2] S_n(4j+r) = \sum_{s=1}^{4k+r-3} \binom{4k+r}{s} c_{4k+r-s} n^s,$$

where the numbers c_j are determined by the following formulas: for $j = 0, 1, 2, \ldots$,

$$c_{4j} = 2(-4)^j - 4, \quad c_{4j+1} = 2(-4)^j - 2, \quad c_{4j+2} = 0, \quad c_{4j+3} = -4(-4)^j - 2.$$

After proving Theorem 1.1, we use it to compute $S_n(5)$. One of the key ideas in all of these results is the generating function $(e^x-1)(e^{\theta x}-1)\cdots(e^{\theta^{m-1}x}-1)$, where θ is any primitive m^{th} root of unity. This generating function, an interesting topic in its own right, is discussed in Section 2.

We also prove similar formulas for the alternating sums

$$T_n(k) = 1^k - 2^k + 3^k - \cdots + (-1)^{n-1}n^k, \tag{1.5}$$

and, finally, we show how the results of this paper can be applied to the Bernoulli and Genocchi numbers.

2. GENERATING FUNCTIONS

Let θ be a primitive m^{th} root of unity so that $\theta^m = 1$ and $\theta^h \neq 1$ for $0 < h < m$. For example, we could let $\theta = e^{2\pi i/m}$.

Define the numbers b_j and c_j by means of the generating functions

$$\prod_{u=0}^{m-1}(e^{\theta^u x}-1) = (e^x-1)(e^{\theta x}-1)\cdots(e^{\theta^{m-1}x}-1) = \sum_{j=0}^{\infty} b_j \frac{x^j}{j!}, \tag{2.1}$$

and

$$e^x \prod_{u=1}^{m-1}(e^{\theta^u x}-1) = \sum_{j=0}^{\infty} c_j \frac{x^j}{j!}. \tag{2.2}$$

Note that any primitive m^{th} root of unity can be used in (2.1) and (2.2). The numbers b_j and c_j depend on m, but the value of m will always be clear when we use this notation. Note also that $b_0 = 0$ and for $m = 1$, we have $c_j = 1$.

If we replace x by $-x$ in (2.2), we have

$$\begin{aligned}\sum_{j=0}^{\infty}(-1)^j c_j \frac{x^j}{j!} &= e^{-x}\prod_{u=1}^{m-1}(e^{-\theta^u x}-1) = \prod_{u=1}^{m-1}(e^{\theta^u x}-1)/[(-1)^{m-1}e^{x(1+\theta+\cdots+\theta^{m-1})}] \\ &= (-1)^{m-1}\prod_{u=1}^{m-1}(e^{\theta^u x}-1).\end{aligned}$$

This gives us another useful generating function for c_j:

$$\prod_{u=1}^{m-1}(e^{\theta^u x}-1) = \sum_{j=0}^{\infty}(-1)^{m+j-1}c_j \frac{x^j}{j!}. \tag{2.3}$$

From (2.1), (2.2), and (2.3), we have

$$\begin{aligned}\sum_{j=0}^{\infty} c_j \frac{x^j}{j!} &= e^x\prod_{u=1}^{m-1}(e^{\theta^u x}-1) = \prod_{u=0}^{m-1}(e^{\theta^u x}-1) + \prod_{u=1}^{m-1}(e^{\theta^u x}-1) \\ &= \sum_{j=0}^{\infty} b_j \frac{x^j}{j!} + \sum_{j=0}^{\infty}(-1)^{m+j-1}c_j \frac{x^j}{j!}.\end{aligned} \tag{2.4}$$

Thus, we have $b_j = (1+(-1)^{m+j})c_j$; that is,

$$b_j = \begin{cases} 2c_j & \text{if } (m+j) \text{ is even,} \\ 0 & \text{if } (m+j) \text{ is odd.}\end{cases} \tag{2.5}$$

To prove the main result of this section, Theorem 2.1, we need the following lemma.

Lemma 2.1 (multisection of series): Let θ be any primitive m^{th} root of unity (such as $e^{2\pi i/m}$) and let $F(x)=\sum_{k=0}^{\infty}a_k x^k$ for complex numbers a_k. Then, for $r=0,1,\dots,m-1$,

$$\sum_{j=0}^{\infty}a_{mj+r}x^{mj+r}=\frac{1}{m}\sum_{j=0}^{m-1}\theta^{(m-j)r}F(\theta^j x). \tag{2.6}$$

If z_0 is a complex number in the circle of convergence of $F(x)$, we can replace x by z_0 in (2.6). Multisection is discussed in [8, p. 131], and a proof of Lemma 2.1 is given in [3].

Theorem 2.1: Let θ be a primitive m^{th} root of unity, and let b_j be defined by (2.1). Then $b_j=0$ unless j is a multiple of m. Furthermore, if m is odd then $b_j=0$ unless j is an *odd* multiple of m.

Proof: We take the logarithm of both sides of (2.1) to obtain

$$\log(e^x-1)+\log(e^{\theta x}-1)+\cdots+\log(e^{\theta^{m-1}x}-1)=\log\sum_{j=0}^{\infty}b_j\frac{x^j}{j!}. \tag{2.7}$$

In (2.6), let $F(x)=\log(e^x-1)$ and $r=0$, and compare the left side of (2.7) with the right side of (2.6) to obtain

$$m\sum_{j=0}^{\infty}a_{mj}x^{mj}=\log\sum_{j=0}^{\infty}b_j\frac{x^j}{j!}. \tag{2.8}$$

Applying the exponential function to (2.8), we have

$$\exp\left(m\sum_{j=0}^{\infty}a_{mj}x^{mj}\right)=\sum_{j=0}^{\infty}b_j\frac{x^j}{j!}. \tag{2.9}$$

We now compare coefficients of x^j on both sides of (2.9) and see that $b_j=0$ unless j is a multiple of m. Now suppose m is odd. Replacing x by $-x$ in (2.1), we have

$$(e^{-x}-1)(e^{-\theta x}-1)\cdots(e^{-\theta^{m-1}x}-1)=\sum_{j=0}^{\infty}(-1)^{mj}b_{mj}\frac{x^{mj}}{(mj)!}.$$

Thus,

$$\prod_{u=0}^{m-1}(e^{-ux}-1)+\prod_{u=0}^{m-1}(e^{ux}-1)=2\sum_{j=0}^{\infty}b_{2mj}\frac{x^{2mj}}{(2mj)!}. \tag{2.10}$$

Now we observe that

$$\prod_{u=0}^{m-1}(e^{-\theta^u x}-1)=\prod_{u=0}^{m-1}(e^{\theta^u x}-1)/[(-1)^m e^{x(1+\theta+\cdots+\theta^{m-1})}]=-\prod_{u=0}^{m-1}(e^{\theta^u x}-1).$$

Thus, the left side of (2.10) is equal to 0 and, therefore, $b_{2mj}=0$ for $j\ge 0$. This completes the proof. □

Theorem 2.1 tells us that the generating function (2.1) could be written as:

$$\prod_{u=0}^{m-1}(e^{\theta^u x}-1)=\sum_{j=0}^{\infty}b_{mj}\frac{x^{mj}}{(mj)!}\quad (m\text{ even}); \tag{2.11}$$

$$\prod_{u=0}^{m-1}(e^{\theta^u x}-1)=\sum_{j=0}^{\infty} b_{m(2j+1)}\frac{x^{m(2j+1)}}{(m(2j+1))!} \quad (m \text{ odd}). \tag{2.12}$$

3. A GENERAL FORMULA

We are now ready to prove our main result, a general lacunary recurrence for the sums $S_n(mj+r)$. We will need the following generating function:

$$F_1(x)=\frac{e^{(1+n)x}-e^x}{e^x-1}=e^x+e^{2x}+\cdots+e^{nx}=\sum_{j=0}^{\infty}S_n(j)\frac{x^j}{j!}. \tag{3.1}$$

Theorem 3.1: Let $S_n(j)$, b_j, and c_j be defined by (1.1), (2.1), and (2.2), respectively. If m is a positive integer, then, for $r=0,1,\ldots,m-1$:

$$\sum_{j=0}^{k-1}\binom{mk+r}{mj+r}b_{(k-j)m}S_n(mj+r)=\sum_{s=1}^{(k-1)m+r+1}\binom{mk+r}{s}c_{mk+r-s}n^s. \tag{3.2}$$

If m is odd and $k\ge 1$, then, for $r=0,1,\ldots,2m-1$:

$$\sum_{j=0}^{k-1}\binom{(2k-1)m+r}{2mj+r}b_{(2k-1-2j)m}S_n(2mj+r)=\sum_{s=1}^{2m(k-1)+r+1}\binom{(2k-1)m+r}{s}c_{(2k-1)m+r-s}n^s. \tag{3.3}$$

Proof: Let $F_1(x)$ be defined by (3.1). We multiply both sides of (2.1) by $F_1(x)$ to obtain

$$(e^{nx}-1)e^x\prod_{u=1}^{m-1}(e^{\theta^u x}-1)=F_1(x)\sum_{j=0}^{\infty}b_j\frac{x^j}{j!}. \tag{3.4}$$

Recalling (2.2) and (2.11), we compare coefficients of x^j on both sides of (3.4) to derive (3.2) for m even or odd.

If m is odd, then by (2.12) we can let $k-j$ be odd in (3.2). We now consider the cases of k even and k odd to obtain (3.3).

Case 1: *k* is even. In (3.2), since $k-j$ is odd, replace k by $2k$ and j by $2j+1$ to obtain

$$\begin{aligned}&\sum_{j=0}^{k-1}\binom{(2k-1)m+(m+r)}{2mj+(m+r)}b_{(2k-1-2j)m}S_n(2mj+m+r)\\&=\sum_{s=1}^{2m(k-1)+m+r+1}\binom{(2k-1)m+(m+r)}{s}c_{(2k-1)m+(m+r)-s}n^s.\end{aligned} \tag{3.5}$$

If we let $r'=(m+r)$;in (3.5), we get (3.3) with r replaced by r' and $m\le r'<2m$.

Case 2: *k* is odd. In (3.2), replace k by $2k-1$ and replace j by $2j$ to obtain (3.3) with $0\le r<m$.

Combining the two cases gives us (3.3) with $0\le r<2m$. This completes the proof. □

We illustrate Theorem 3.1 by proving formulas (1.2) and (1.3) and Theorem 1.1.

Let $m=2$. From definitions (2.1) and (2.2), we see that $b_{2j+1}=0$ and, for $j>0$, $b_{2j}=-2$ and $c_j=-1$. Thus,

$$2\sum_{j=0}^{k-1}\binom{2k+r}{2j+r}S_n(2j+r)=\sum_{s=1}^{2k+r-1}\binom{2k+r}{s}n^s,$$

which is equivalent to (1.2) and (1.3).

Formula (1.4) can be deduced from Theorem 3.1 by letting $m=3$.

To prove Theorem 1.1, we let $m=4$ and $\theta=i$. The left side of

$$e^x(e^{ix}-1)(e^{i^2x}-1)(e^{i^3x}-1)=\sum_{j=0}^{\infty}c_j\frac{x^j}{j!}$$

can be written

$$2-2e^x-e^{ix}-e^{-ix}-e^{(1+i)x}-e^{(1-i)x},$$

so for $j>0$,

$$c_j=-2-i^j-(-i)^j+(1+i)^j+(1-i)^j.$$

This gives us

$$c_{4j}=-4+2(-4)^j,\quad c_{4j+1}=-2+2(-4)^j,\quad c_{4j+2}=0,\quad c_{4j+3}=-2-4(-4)^j. \tag{3.6}$$

By (2.5) we have, for $j\geq 1$,

$$b_{4j}=2c_{4j}=-8+4(-4)^j. \tag{3.7}$$

For $m=4$ and the values of b_j and c_j given by (3.6) and (3.7), equation (3.2) gives Theorem 1.1. This completes the proof. □

To illustrate Theorem 1.1, we compute $S_n(5)$. In Theorem 1.1, let $r=1$ and $k=2$ to obtain

$$-24\binom{9}{5}S_n(5)=-56\binom{9}{1}S_n(1)+\sum_{s=1}^{6}\binom{9}{s}c_{9-s}n^s.$$

Using (3.6) and the formula $S_n(1)=n(n+1)/2$, we have

$$S_n(5)=-\frac{1}{12}n^2+\frac{5}{12}n^4+\frac{1}{2}n^5+\frac{1}{6}n^6.$$

We could easily keep going here and compute $S_n(9)$, $S_n(13)$, and so on.

4. ALTERNATING SUMS

The methods of Sections 2 and 3 can be used just as easily on the alternating sums $T_n(k)$ defined by (1.5). Let θ be any primitive m^{th} root of unity, and define the numbers g_j and h_j by means of the generating functions

$$\prod_{u=0}^{m-1}(e^{\theta^u x}+1)=(e^x+1)(e^{\theta x}+1)\cdots(e^{\theta^{m-1}x}+1)=\sum_{j=0}^{\infty}g_j\frac{x^j}{j!}, \tag{4.1}$$

and

$$e^x\prod_{u=1}^{m-1}(e^{\theta^u x}+1)=\sum_{j=0}^{\infty}h_j\frac{x^j}{j!}. \tag{4.2}$$

Note that g_j and h_j are functions of m.

Analogous to (2.3), and proved in the same way, is another generating function for h_j:

$$\prod_{u=1}^{m-1}(e^{\theta^u x}+1)=\sum_{j=0}^{\infty}(-1)^j h_j \frac{x^j}{j!}. \tag{4.3}$$

Equations (4.1), (4.2), and (4.3) give us the relationship $h_j=\frac{1}{2}g_j$ if j is even.

Theorems 4.1 and 4.2 are analogous to Theorems 2.1 and 3.1, and they are proved in exactly the same say. The following generating function is used in the proof of Theorem 4.2:

$$F_2(x)=\frac{(-1)^{1+n}e^{(1+n)x}+e^x}{e^x+1}=e^x-e^{2x}+\cdots+(-1)^{1+n}e^{nx}=\sum_{j=0}^{\infty}T_n(j)\frac{x^j}{j!}. \tag{4.4}$$

Theorem 4.1: Let θ be a primitive m^{th} root of unity and let g_j be defined by (4.1). Then $g_j=0$ unless j is a multiple of m. Furthermore, if m is odd, then $g_j=0$ unless j is an *even* multiple of m.

Theorem 4.2: Let $T_n(j)$ be defined by (1.5) and let g_j and h_j be defined by (4.1) and (4.2), respectively. If m is a positive integer, then, for $r=0,1,\dots,m-1$,

$$\sum_{j=0}^{k}\binom{mk+r}{mj+r}g_{(k-j)m}T_n(mj+r)=(-1)^{n+1}\sum_{s=0}^{mk+r}\binom{mk+r}{s}h_{mk+r-s}n^s+h_{mk+r}. \tag{4.5}$$

If m is odd, then, for $r=0,1,\dots,2m-1$,

$$\sum_{j=0}^{k}\binom{2mk+r}{2mj+r}g_{(k-j)2m}T_n(2mj+r)=(-1)^{n+1}\sum_{s=0}^{2mk+r}\binom{2mk+r}{s}h_{2mk+r-s}n^s+h_{2mk+r}. \tag{4.6}$$

We note that, by (4.2), $h_0=2^{m-1}$, so the right sides of (4.5) and (4.6) are polynomials in n of degrees $mk+r$ and $2mk+r$, respectively. We also note that $g_0=2^m$.

For example, let $m=2$ and $\theta=-1$. Then we have $g_0=4$, $g_{2j}=2$ if $j>0$, $h_0=2$, and $h_j=1$ if $j>0$. Theorem 4.2 gives us

$$\begin{aligned}4T_n(2k+r)=&-2\sum_{j=0}^{k-1}\binom{2k+r}{2j+r}T_n(2j+r)+1+(-1)^{n+1}\\&+(-1)^{n+1}\left\{\sum_{s=1}^{2k+r-1}\binom{2k+r}{s}n^s+2n^{2k+r}\right\}.\end{aligned}$$

The following formula for $m=3$, which is analogous to (1.4), was given in [4]. Let $m=3$, let θ be a primitive third root of unity, and let w_j be defined as in Section 1. Then, for $n>0$ and $k\geq 0$, with r and k not both 0,

$$\begin{aligned}8T_n(6k+r)=&-6\sum_{j=0}^{k-1}\binom{6k+r}{6j+r}T_n(6j+r)+[1+(-1)^{n+1}]w_r\\&+(-1)^{n+1}\left\{\sum_{s=1}^{6k+r-1}\binom{6k+r}{s}w_{r-s}n^s+4n^{6k+r}\right\}.\end{aligned}$$

If $m=4$, we use Theorem 4.2 to prove the following new result.

Theorem 4.3: Let $n > 0$ and $k \geq 0$. Then, for $r = 0, 1, 2, 3,$ and r and k not both 0,

$$16T_n(4k+r) = -4\sum_{j=0}^{k-1}\binom{4k+r}{4j+r}[(-4)^{k-j}+2]T_n(4j+r) + h_{4k+r}$$
$$+(-1)^{n+1}\sum_{s=0}^{4k+r}\binom{4k+r}{s}h_{4k+r-s}n^s,$$

where $h_0 = 8$ and the numbers h_j, except for h_0, are determined by the following formulas: for $j = 0, 1, 2, \ldots,$

$$h_{4j} = 2(-4)^{k-h}+4, \quad h_{4j+1} = 2(-4)^{k-h}+2, \quad h_{4j+2} = 0, \quad h_{4j+3} = -4(-4)^{k-h}+2.$$

5. BERNOULLI AND GENOCCHI NUMBERS

The methods of this paper can be applied to other special number sequences. For example, consider the Bernoulli numbers B_n defined by the generating function

$$\frac{x}{e^x-1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}. \tag{5.1}$$

These numbers are well known and have been studied extensively (see, e.g., [7, ch. 2]). It is well known that $B_0 = 1$, $B_1 = -\frac{1}{2}$, and $B_{2k+1} = 0$ for $k > 0$.

We can use the methods of this paper to derive the following general lacunary recurrence for the Bernoulli numbers.

Theorem 5.1: Let B_n be defined by (5.1) and let b_j and c_j be defined by (2.1) and (2.2), respectively. If m is a positive integer and $k > 0$, then, for r even, $0 \leq r < m$,

$$\sum_{j=0}^{k-1}\binom{mk+r}{mj+r}b_{(k-j)m}B_{mj+r} = (mk+r)c_{mk+r-1}.$$

If m is odd and $k > 0$, then, for r even, $0 \leq r < 2m$,

$$\sum_{j=0}^{k-1}\binom{(2k-1)m+r}{2mj+r}b_{(2k-1-2j)m}B_{2mj+r} = [(2k-1)m+r]c_{(2k-1)m+r-1}.$$

Proof: Multiply both sides of (2.1) by $x/(e^x-1)$ to obtain

$$x\prod_{u=1}^{m-1}(e^{\theta^u x}-1) = \frac{x}{e^x-1}\sum_{j=0}^{\infty}b_j\frac{x^j}{j!}. \tag{5.2}$$

By (2.3) and (5.2) we have, for $n > 0$,

$$(-1)^{m+n}nc_{n-1} = \sum_{j=0}^{n}\binom{n}{j}B_j b_{n-j}.$$

The remainder of the proof is similar to the proof of Theorem 3.1. □

Several writers, like Chellali [1], Lehmer [5], Ramanujan [7], and Riordan [8, pp. 136-40] have developed lacunary formulas for the Bernoulli numbers (see [2] for references).

Obviously, the methods of this paper can also be used on the Genocchi numbers G_n, which are defined by

$$\frac{2x}{e^x+1} = \sum_{n=1}^{\infty} G_n \frac{x^n}{n!}.$$

Lacunary recurrences for the Genocchi numbers can be found in [2]. Incidentally, it is well known that the Genocchi numbers are integers and that $G_{2j} = 2(1-2^{2j})B_{2j}$.

As a final comment, we note that the numbers b_j and g_j of this paper are special cases of the generalized Bernoulli and Euler numbers of Nörlund [6, pp. 142-43], which are defined by

$$(e^{\omega_1 x}-1)(e^{\omega_2 x}-1)\cdots(e^{\omega_m x}-1) = (\omega_1\omega_2\ldots\omega_m)\sum_{j=0}^{\infty} B_j^{(-m)}(\omega_1,\ldots,\omega_m)\frac{x^{j+m}}{j!}$$

and

$$(e^{\omega_1 x}+1)(e^{\omega_2 x}+1)\cdots(e^{\omega_m x}+1) = \sum_{j=0}^{\infty} 2^{m-j} C_j^{(-m)}(\omega_1,\ldots,\omega_m)\frac{x^j}{j!},$$

where $\omega_1,\ldots,\omega_m$ are arbitrary complex numbers. To the writer's knowledge, none of the properties of b_n and g_n developed in this paper were proved by Nörlund.

REFERENCES

1. M. Chellali. "Accélération de calcul de nombres de Bernoulli." *J. Number Theory* **28** (1988): 347-62.
2. F. T. Howard. "Formulas of Ramanujan Involving Lucas Numbers, Pell Numbers, and Bernoulli Numbers." In *Applications of Fibonacci Numbers* **6**:257-70. Dordrecht: Kluwer, 1996.
3. F. T. Howard & Richard Witt. "Lacunary Sums of Binomial Coefficients." In *Applications of Fibonacci Numbers* **7**:185-95. Dordrecht: Kluwer, 1998.
4. F. T. Howard. "Sums of Powers of Integers via Generating Functions." *The Fibonacci Quarterly* **34.3** (1996):244-56.
5. D. H. Lehmer. "Lacunary Recurrence Formulas for the Numbers of Bernoulli and Euler." *Annals of Math.* **36** (1935):637-49.
6. N. Nörlund. *Vorlesungen über Differenzenrechnumg.* New York: Chelsea, 1954.
7. S. Ramanujan. "Some Properties of Bernoulli's Numbers." *J. Indian Math. Soc.* **3** (1911): 219-34.
8. J. Riordan. *Combinatorial Identities.* New York: Wiley, 1968.

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