GENERALIZED TRIPLE PRODUCTS

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1. INTRODUCTION

For arbitrary integers a and b, Horadam [2] and [3] established the notation

$$W_n = W_n(a, b; p, q), \tag{1.1}$$

meaning that

$$W_n = pW_{n-1} - qW_{n-2},\ W_0 = a,\ W_1 = b,\quad n \geq 2. \tag{1.2}$$

The sequence $\{W_n\}_{n=0}^{\infty}$ thus defined can be extended to negative integer subscripts by the use of (1.2), and with this understanding we write simply $\{W_n\}$. In this paper we assume that a, b, p, and q are arbitrary real numbers.

By using the generating functions of $\{F_{n+m}\}_{n=0}^{\infty}$ and $\{L_{n+m}\}_{n=0}^{\infty}$ Hansen [1] obtained expansions for $F_jF_kF_l$, $F_jF_kL_l$, $F_jL_kL_l$, and $L_jL_kL_l$. By following the same techniques, Serkland [5] produced similar expansions for the Pell and Pell-Lucas numbers defined by

$$\begin{cases} P_n = W_n(0, 1; 2, -1), \\ Q_n = W_n(2, 2; 2, -1). \end{cases} \tag{1.3}$$

Later Horadam [4] generalized the results of both these writers to the sequences

$$\begin{cases} U_n = W_n(0, 1; p, -1), \\ V_n = W_n(2, p; p, -1). \end{cases} \tag{1.4}$$

Define the sequences $\{W_n\}$ and $\{X_n\}$ by

$$\begin{cases} W_n = W_n(a, b; p, -1), \\ X_n = W_{n+1} + W_{n-1}. \end{cases} \tag{1.5}$$

Here we emphasize that W_n is as in (1.2) but with $q = -1$, and this is the case for the remainder of the paper. Since $\{W_n\}$ generalizes $\{U_n\}$, then $\{X_n\}$ generalizes $\{V_n\}$ by virtue of the fact that $V_n = U_{n+1} + U_{n-1}$. The object of this paper is to generalize the results of Horadam, and so also of Serkland and of Hansen, by incorporating terms from the sequences $\{W_n\}$ and $\{X_n\}$ into the products.

Since $\Delta = p^2 + 4 \neq 0$, the roots α and β of $x^2 - px - 1 = 0$ are distinct. Hence, the Binet form (see [2] and [3]) for W_n is

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta},$$

where $A = b - a\beta$ and $B = b - a\alpha$. It can also be shown that

$$X_n = A\alpha^n + B\beta^n.$$

2. SOME PRELIMINARY RESULTS

We shall need the following, each of which can be proved with the use of Binet forms:

$$(-1)^n U_{-n} = -U_n, \tag{2.1}$$

$$(-1)^n V_{-n} = V_n, \tag{2.2}$$

$$\Delta W_n = X_{n+1} + X_{n-1}, \tag{2.3}$$

$$\Delta U_{m+d} W_{n-d} - V_m X_n = (-1)^{m+1} V_d X_{n-m-d}, \tag{2.4}$$

$$W_{m+d} V_{n-d} - U_m X_n = (-1)^m W_d V_{n-m-d}, \tag{2.5}$$

$$W_n U_m + W_{n-1} U_{m-1} = W_{n+m-1}, \tag{2.6}$$

$$W_n V_m + W_{n-1} V_{m-1} = X_{n+m-1}, \tag{2.7}$$

$$U_n X_m + U_{n-1} X_{m-1} = X_{n+m-1}, \tag{2.8}$$

$$X_n V_m + X_{n-1} V_{m-1} = X_{n+m} + X_{n+m-2} = \Delta W_{n+m-1}. \tag{2.9}$$

3. THE MAIN RESULTS

Using the Binet form for W_n we have, for m an integer and $|x|$ small,

$$\sum_{n=0}^{\infty} W_{n+m} x^n = \sum_{n=0}^{\infty} \frac{(A\alpha^{n+m} - B\beta^{n+m})x^n}{\alpha - \beta} = \frac{1}{\alpha - \beta}\left(\alpha^m \sum_{n=0}^{\infty} A\alpha^n x^n - \beta^m \sum_{n=0}^{\infty} B\beta^n x^n\right)$$

$$= \frac{1}{\alpha - \beta}\left(\frac{A\alpha^m}{1-\alpha x} - \frac{B\beta^m}{1-\beta x}\right) = \frac{1}{\alpha-\beta}\left(\frac{(A\alpha^m - B\beta^m) - \alpha\beta(A\alpha^{m-1} - B\beta^{m-1})x}{(1-\alpha x)(1-\beta x)}\right).$$

Then, putting $D = 1 - px - x^2$, we have

$$\sum_{n=0}^{\infty} W_{n+m} x^n = \frac{W_m + W_{m-1}x}{D}. \tag{3.1}$$

Of course, in (3.1), we can replace $\{W_n\}$ by any of the sequences in this paper. In particular, with $m = 1$ and $\{W_n\} = \{U_n\}$, (3.1) becomes

$$\sum_{n=0}^{\infty} U_{n+1} x^n = \frac{1}{D}. \tag{3.2}$$

The following result, which is essential for what follows, can be proved with partial fractions techniques:

$$\begin{aligned} \frac{(j+kx)}{D} \cdot \frac{(l+tx)}{D} &= \frac{jl + (jt+kl)x + ktx^2}{D^2} \\ &= \frac{-kt}{D} + \frac{(jl+kt) + (jt+kl-pkt)x}{D^2}. \end{aligned} \tag{3.3}$$

Now

$$\frac{U_m + U_{m-1}x}{D} \cdot \frac{X_s + X_{s-1}x}{D} = \sum_{n=0}^{\infty} U_{n+m} x^n \cdot \sum_{n=0}^{\infty} X_{n+s} x^n = \sum_{n=0}^{\infty} \sum_{i=0}^{n} U_{i+m} X_{n-i+s} x^n. \tag{3.4}$$

Alternatively, using (3.3), we have

$$\frac{U_m+U_{m-1}x}{D}\cdot\frac{X_s+X_{s-1}x}{D}$$
$$=\frac{-U_{m-1}X_{s-1}}{D}+\frac{(U_mX_s+U_{m-1}X_{s-1})+(U_mX_{s-1}+U_{m-1}X_s-pU_{m-1}X_{s-1})x}{D^2}.$$

Then, by using (2.8) and the recurrence relation (1.2), this becomes

$$\frac{-U_{m-1}X_{s-1}}{D}+\frac{X_{m+s-1}+(U_{m-1}X_s+U_{m-2}X_{s-1})x}{D^2}$$
$$=\frac{-U_{m-1}X_{s-1}}{D}+\frac{X_{m+s-1}+X_{m+s-2}x}{D^2}=-(U_{m-1}X_{s-1})\cdot\frac{1}{D}+\frac{X_{m+s-1}+X_{m+s-2}x}{D}\cdot\frac{1}{D}.$$

Now, by using (3.1) and (3.2), this in turn becomes

$$-U_{m-1}X_{s-1}\sum_{n=0}^{\infty}U_{n+1}x^n+\sum_{n=0}^{\infty}X_{n+m+s-1}x^n\cdot\sum_{n=0}^{\infty}U_{n+1}x^n$$
$$=\sum_{n=0}^{\infty}(-U_{n+1}U_{m-1}X_{s-1})x^n+\sum_{n=0}^{\infty}\sum_{i=0}^{n}U_{i+1}X_{n-i+m+s-1}x^n$$
$$=\sum_{n=0}^{\infty}\left(-U_{n+1}U_{m-1}X_{s-1}+\sum_{i=0}^{n}U_{i+1}X_{n-i+m+s-1}\right)x^n.$$

By equating the coefficients of x^n in the last line and the right side of (3.4), we obtain

$$\sum_{i=0}^{n}U_{i+m}X_{n-i+s}=-U_{n+1}U_{m-1}X_{s-1}+\sum_{i=0}^{n}U_{i+1}X_{n-i+m+s-1}.$$

Finally, putting $j=m-1$, $k=n+1$, and $l=s-1$, we get

$$U_jU_kX_l=\sum_{i=0}^{k-1}(U_{i+1}X_{j+k+l-i}-U_{j+i+1}X_{k+l-i}). \tag{3.5}$$

If we replace X by V, we see that this generalizes Horadam's Theorem 4, which contains a typographical error in one of the subscripts.

In exactly the same manner, taking the product of

$$\frac{U_m+U_{m-1}x}{D}\quad\text{and}\quad\frac{W_s+W_{s-1}x}{D}$$

and using (2.6), we obtain

$$W_jU_kU_l=\sum_{i=0}^{l-1}(W_{j+k+l-i}U_{i+1}-W_{j+i+1}U_{k+l-i}). \tag{3.6}$$

This generalizes Horadam's Theorem 5.

Again, taking the product of

$$\frac{V_m+V_{m-1}x}{D}\quad\text{and}\quad\frac{X_s+X_{s-1}x}{D}$$

and using (2.9) yields

$$U_jV_kX_l = \sum_{i=0}^{j-1}(\Delta U_{j-i}W_{k+l+i+1} - V_{k+i+1}X_{j+l-i}). \tag{3.7}$$

This generalizes Horadam's Theorem 6.

Further, taking the product of

$$\frac{W_m + W_{m-1}x}{D} \quad \text{and} \quad \frac{V_s + V_{s-1}x}{D}$$

and using (2.7) leads to

$$W_jU_kV_l = \sum_{i=0}^{k-1}(U_{i+1}X_{j+k+l-i} - W_{j+i+1}V_{k+l-i}). \tag{3.8}$$

Making use of (3.7), we have

$$\begin{aligned} V_jV_kX_l &= (U_{j+1}+U_{j-1})V_kX_l = U_{j+1}V_kX_l + U_{j-1}V_kX_l \\ &= \sum_{i=0}^{j}(\Delta U_{j-i+1}W_{k+l+i+1} - V_{k+i+1}X_{j+l-i+1}) + \sum_{i=0}^{j-2}(\Delta U_{j-i-1}W_{k+l+i+1} - V_{k+i+1}X_{j+l-i-1}) \\ &= \left(\sum_{i=0}^{j-2}(\Delta W_{k+l+i+1}(U_{j-i+1}+U_{j-i-1}) - V_{k+i+1}(X_{j+l-i+1}+X_{j+l-i-1}))\right) \\ &\quad + (\Delta U_2W_{k+l+j} - V_{k+j}X_{l+2}) + (\Delta U_1W_{k+l+j+1} - V_{k+j+1}X_{l+1}). \end{aligned}$$

We now use (2.4) and (2.2) to simplify the last two terms on the right side. Finally, recalling that $U_{n+1}+U_{n-1}=V_n$ and using (2.3), we obtain

$$V_jV_kX_l = \left(\Delta\sum_{i=0}^{j-2}(W_{k+l+i+1}V_{j-i} - W_{j+l-i}V_{k+i+1})\right) + pX_lV_{j+k-1}. \tag{3.9}$$

This generalizes Horadam's Theorem 7, and is more concisely written.

To obtain our final product, we write

$$W_jV_kV_l = W_j(U_{k+1}+U_{k-1})V_l.$$

Then proceeding in the same manner we use (3.8) and (2.5) to obtain

$$W_jV_kV_l = \left(\Delta\sum_{i=0}^{k-2}(W_{j+k+l-i}U_{i+1} - W_{j+i+1}U_{k+l-i})\right) + (-1)^{k+1}pW_jV_{l+1-k}. \tag{3.10}$$

Of course, in each summation identity, the parameter contained in the upper limit of summation must be chosen so that the sum is well defined. For example, in (3.10), we assume $k \ge 2$.

4. THE MAIN RESULTS SIMPLIFIED

We have chosen to present the results (3.5)-(3.10) in the given manner in order to facilitate comparison with the results of Horadam, Serkland, and Hansen. We now demonstrate that they can be simplified considerably.

By using Binet forms, it can be shown that

$$U_{i+1}X_{j+k+l-i} - U_{j+i+1}X_{k+l-i} = (-1)^i U_j X_{k+l-2i-1}, \tag{4.1}$$

$$W_{j+k+l-i}U_{i+1} - W_{j+i+1}U_{k+l-i} = (-1)^i W_j U_{k+l-2i-1}, \tag{4.2}$$

$$\Delta U_{j-i}W_{k+l+i+1} - V_{k+i+1}X_{j+l-i} = (-1)^{i+j+1} X_l V_{k+2i+1-j}, \tag{4.3}$$

$$U_{i+1}X_{j+k+l-i} - W_{j+i+1}V_{k+l-i} = (-1)^i W_j V_{k+l-2i-1}, \tag{4.4}$$

$$W_{k+l+i+1}V_{j-i} - W_{j+l-i}V_{k+i+1} = (-1)^{i+j} X_l U_{k+2i+1-j}, \tag{4.5}$$

$$W_{j+k+l-i}U_{i+1} - W_{j+i+1}U_{k+l-i} = (-1)^i W_j U_{k+l-2i-1}. \tag{4.6}$$

Now, if we substitute the left side of (4.1) into (3.5) and replace k by j and l by k, we obtain

$$U_j X_k = \sum_{i=0}^{j-1} (-1)^i X_{j+k-2i-1}. \tag{4.7}$$

In the same manner, we use (4.2)-(4.6) to simplify (3.6)-(3.10), which become, respectively,

$$U_j U_k = \sum_{i=0}^{k-1} (-1)^i U_{j+k-2i-1}, \tag{4.8}$$

$$U_j V_k = \sum_{i=0}^{j-1} (-1)^{i+j+1} V_{k+2i+1-j}, \tag{4.9}$$

$$U_j V_k = \sum_{i=0}^{j-1} (-1)^i V_{j+k-2i-1}, \tag{4.10}$$

$$V_j V_k = \left(\Delta \sum_{i=0}^{j-2} (-1)^{i+j} U_{k+2i+1-j} \right) + pV_{j+k-1}, \tag{4.11}$$

$$V_j V_k = \left(\Delta \sum_{i=0}^{j-2} (-1)^i U_{j+k-2i-1} \right) + (-1)^{j+1} pV_{k+1-j}. \tag{4.12}$$

By noting that $\sum_{i=0}^n f(i) = \sum_{i=0}^n f(n-i)$, we see that the right sides of (4.9) and (4.10) are identical. However, the right sides of (4.11) and (4.12) are different expressions which reduce to $V_j V_k$.

REFERENCES

1. R. T. Hansen. "Generating Identities for Fibonacci and Lucas Triples." *The Fibonacci Quarterly* **10.6** (1972):571-78.
2. A. F. Horadam. "Basic Properties of a Certain Generalized Sequence of Numbers." *The Fibonacci Quarterly* **3.4** (1965):161-76.
3. A. F. Horadam. "Generating Functions for Powers of a Certain Generalized Sequence of Numbers." *Duke Math. J.* **32** (1965):437-46.
4. A. F. Horadam. "Generating Identities for Generalized Fibonacci and Lucas Triples." *The Fibonacci Quarterly* **15.4** (1977):289-92.
5. C. Serkland. "Generating Identities for Pell Triples." *The Fibonacci Quarterly* **12.2** (1974): 121-28.

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