LUCAS SEQUENCES AND FUNCTIONS OF A 3-BY-3 MATRIX

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1. INTRODUCTION

Define the sequences $\{U_n\}$ and $\{V_n\}$ for all integers n by

$$\begin{cases} U_n = pU_{n-1} - qU_{n-2}, & U_0 = 0,\ U_1 = 1, \\ V_n = pV_{n-1} - qV_{n-2}, & V_0 = 2,\ V_1 = p, \end{cases} \tag{1.1}$$

where p and q are real numbers with $q(p^2-4q) \neq 0$. These sequences were studied originally by Lucas [9], and have subsequently been the subject of much attention.

The Binet forms for U_n and V_n are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n,$$

where

$$\alpha = \frac{p + \sqrt{p^2 - 4q}}{2} \quad \text{and} \quad \beta = \frac{p - \sqrt{p^2 - 4q}}{2}$$

are the roots, assumed distinct, of $x^2 - px + q = 0$. We assume further that α / β is not an n^{th} root of unity for any n. Write

$$\Delta = (\alpha - \beta)^2 = p^2 - 4q.$$

A well-known relationship between U_n and V_n is

$$V_n = U_{n+1} - qU_{n-1}, \tag{1.2}$$

which we use subsequently.

Barakat [2] considered the matrix exponential, $\exp(X)$, for the 2-by-2 matrix

$$X = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

where he took $\text{trace}(X) = p$ and $\det(X) = q$. In so doing, he established various infinite sums involving terms from $\{U_n\}$ and $\{V_n\}$.

Following Barakat, Walton [13] evaluated the series for the sine and cosine functions at the matrix X and obtained further infinite sums involving terms from $\{U_n\}$ and $\{V_n\}$. Extensions of these ideas to higher-order recurrences have been given by Shannon and Horadam [12] and Pethe [11]. Recently, many papers have appeared which have followed the theme of these writers. See, for example, Brugia and Filipponi [5], Filipponi and Horadam [6], Horadam and Filipponi [8], and Melham and Shannon [10].

In this paper we apply the techniques of the above writers to a 3-by-3 matrix to obtain new infinite sums involving *squares* of terms from the sequences $\{U_n\}$ and $\{V_n\}$.

2. THE MATRIX $R_{k,x}$

Berzsenyi [4] has shown that the matrix

$$R = \begin{pmatrix} 0 & 0 & q^2 \\ 0 & -q & -2pq \\ 1 & p & p^2 \end{pmatrix} \tag{2.1}$$

is such that, for nonnegative integers n,

$$R^n = \begin{pmatrix} q^2U_{n-1}^2 & q^2U_{n-1}U_n & q^2U_n^2 \\ -2qU_{n-1}U_n & -q(U_n^2+U_{n-1}U_{n+1}) & -2qU_nU_{n+1} \\ U_n^2 & U_nU_{n+1} & U_{n+1}^2 \end{pmatrix}. \tag{2.2}$$

The characteristic equation of R is

$$\lambda^3 + (q-p^2)\lambda^2 + q(p^2-q)\lambda - q^3 = 0. \tag{2.3}$$

Since $p = \alpha + \beta$ and $q = \alpha\beta$, it is readily verified that α^2, β^2, and $\alpha\beta$ are the eigenvalues of R. These eigenvalues are nonzero and distinct because of our assumptions in Section 1.

Associated with R, we define the matrix $R_{k,x}$ by

$$R_{k,x} = xR^k = x\begin{pmatrix} q^2U_{k-1}^2 & q^2U_{k-1}U_k & q^2U_k^2 \\ -2qU_{k-1}U_k & -q(U_k^2+U_{k-1}U_{k+1}) & -2qU_kU_{k+1} \\ U_k^2 & U_kU_{k+1} & U_{k+1}^2 \end{pmatrix}, \tag{2.4}$$

where x is an arbitrary real number and k is a nonnegative integer. From the definition of an eigenvalue, it follows immediately that $x\alpha^{2k}, x\beta^{2k}$, and xq^k are the eigenvalues of $R_{k,x}$. Again, they are nonzero and distinct.

3. THE MAIN RESULTS

Suppose $f(z) = \Sigma_{n=0}^{\infty} a_n z^n$ is a power series whose domain of convergence includes $x\alpha^{2k}$, $x\beta^{2k}$, and xq^k. Then we have, from (2.4),

$$f(R_{k,x}) = \sum_{n=0}^{\infty} a_n R_{k,x}^n = \sum_{n=0}^{\infty} a_n x^n R^{kn}$$

$$= \begin{pmatrix} q^2\sum_{n=0}^{\infty} a_n x^n U_{kn-1}^2 & q^2\sum_{n=0}^{\infty} a_n x^n U_{kn-1}U_{kn} & q^2\sum_{n=0}^{\infty} a_n x^n U_{kn}^2 \\ -2q\sum_{n=0}^{\infty} a_n x^n U_{kn-1}U_{kn} & -q\sum_{n=0}^{\infty} a_n x^n (U_{kn}^2+U_{kn-1}U_{kn+1}) & -2q\sum_{n=0}^{\infty} a_n x^n U_{kn}U_{kn+1} \\ \sum_{n=0}^{\infty} a_n x^n U_{kn}^2 & \sum_{n=0}^{\infty} a_n x^n U_{kn}U_{kn+1} & \sum_{n=0}^{\infty} a_n x^n U_{kn+1}^2 \end{pmatrix}. \tag{3.1}$$

On the other hand, from the theory of matrices ([3] and [7]), it is known that

$$f(R_{k,x}) = c_0I + c_1R_{k,x} + c_2R_{k,x}^2, \tag{3.2}$$

where I is the 3-by-3 identity matrix, and where c_0, c_1, and c_2 can be obtained by solving the system

$$\begin{cases} c_0 + c_1 x\alpha^{2k} + c_2 x^2 \alpha^{4k} = f(x\alpha^{2k}), \\ c_0 + c_1 x\beta^{2k} + c_2 x^2 \beta^{4k} = f(x\beta^{2k}), \\ c_0 + c_1 x\alpha^k \beta^k + c_2 x^2 \alpha^{2k} \beta^{2k} = f(x\alpha^k \beta^k). \end{cases}$$

If we use Cramer's rule and observe, using the Binet form for U_n, that

$$(\alpha^{2k} - \beta^{2k})(\beta^{2k} - \alpha^k \beta^k)(\alpha^k \beta^k - \alpha^{2k}) = q^k U_{2k} U_k^2 (\alpha - \beta)^3,$$

we obtain

$$\begin{cases} c_0 = \dfrac{f(x\alpha^{2k}) q^k \beta^{3k} (\alpha^k - \beta^k) + f(x\beta^{2k}) q^k \alpha^{3k} (\alpha^k - \beta^k) + f(xq^k) q^{2k} (\beta^{2k} - \alpha^{2k})}{q^k U_{2k} U_k^2 (\alpha - \beta)^3}, \\ c_1 = \dfrac{f(x\alpha^{2k}) \beta^{2k} (\beta^{2k} - \alpha^{2k}) + f(x\beta^{2k}) \alpha^{2k} (\beta^{2k} - \alpha^{2k}) + f(xq^k)(\alpha^{4k} - \beta^{4k})}{xq^k U_{2k} U_k^2 (\alpha - \beta)^3}, \\ c_2 = \dfrac{f(x\alpha^{2k}) \beta^k (\alpha^k - \beta^k) + f(x\beta^{2k}) \alpha^k (\alpha^k - \beta^k) + f(xq^k)(\beta^{2k} - \alpha^{2k})}{x^2 q^k U_{2k} U_k^2 (\alpha - \beta)^3}. \end{cases}$$

Now, equating lower left entries in (3.1) and (3.2), we obtain

$$\sum_{n=0}^{\infty} a_n x^n U_{kn}^2 = c_1 x U_k^2 + c_2 x^2 U_{2k}^2. \tag{3.3}$$

With the values of c_1 and c_2 obtained above, the right side of (3.3) is

$$\frac{f(x\alpha^{2k})(\beta^{2k}(\beta^{2k} - \alpha^{2k}) U_k^2 + \beta^k (\alpha^k - \beta^k) U_{2k}^2)}{q^k U_{2k} U_k^2 (\alpha - \beta)^3}$$

$$+ \frac{f(x\beta^{2k})(\alpha^{2k}(\beta^{2k} - \alpha^{2k}) U_k^2 + \alpha^k (\alpha^k - \beta^k) U_{2k}^2)}{q^k U_{2k} U_k^2 (\alpha - \beta)^3}$$

$$+ \frac{f(xq^k)((\alpha^{4k} - \beta^{4k}) U_k^2 + (\beta^{2k} - \alpha^{2k}) U_{2k}^2)}{q^k U_{2k} U_k^2 (\alpha - \beta)^3}.$$

If we note that $U_{2k} = U_k V_k$ and use Binet forms, we obtain finally

$$\sum_{n=0}^{\infty} a_n x^n U_{kn}^2 = \frac{f(x\alpha^{2k}) + f(x\beta^{2k}) - 2f(xq^k)}{\Delta}. \tag{3.4}$$

In precisely the same manner, we equate appropriate entries in (3.1) and (3.2) to obtain

$$\sum_{n=0}^{\infty} a_n x^n U_{kn+1}^2 = \frac{\alpha^2 f(x\alpha^{2k}) + \beta^2 f(x\beta^{2k}) - 2qf(xq^k)}{\Delta}, \tag{3.5}$$

$$\sum_{n=0}^{\infty} a_n x^n U_{kn} U_{kn-1} = \frac{\alpha^{-1} f(x\alpha^{2k}) + \beta^{-1} f(x\beta^{2k}) - \frac{p}{q} f(xq^k)}{\Delta}, \tag{3.6}$$

$$\sum_{n=0}^{\infty} a_n x^n U_{kn} U_{kn+1} = \frac{\alpha f(x\alpha^{2k}) + \beta f(x\beta^{2k}) - pf(xq^k)}{\Delta}, \tag{3.7}$$

$$\sum_{n=0}^{\infty} a_n x^n U_{kn-1}^2 = \frac{\beta^2 f(x\alpha^{2k}) + \alpha^2 f(x\beta^{2k}) - 2qf(xq^k)}{q^2\Delta}, \tag{3.8}$$

$$\sum_{n=0}^{\infty} a_n x^n (U_{kn}^2 + U_{kn-1}U_{kn+1}) = \frac{2f(x\alpha^{2k}) + 2f(x\beta^{2k}) - \frac{p^2}{q} f(xq^k)}{\Delta}. \tag{3.9}$$

From (3.4) and (3.9), we obtain

$$\sum_{n=0}^{\infty} a_n x^n U_{kn-1} U_{kn+1} = \frac{f(x\alpha^{2k}) + f(x\beta^{2k}) + \left(2 - \frac{p^2}{q}\right) f(xq^k)}{\Delta}. \tag{3.10}$$

Finally, from (1.2), we have

$$V_{kn}^2 = U_{kn+1}^2 + q^2 U_{kn-1}^2 - 2qU_{kn+1}U_{kn-1}.$$

This, together with (3.5), (3.8), and (3.10), yields

$$\sum_{n=0}^{\infty} a_n x^n V_{kn}^2 = f(x\alpha^{2k}) + f(x\beta^{2k}) + 2f(xq^k). \tag{3.11}$$

In contrast to our approach, Brugia and Filipponi [5] used the Kronecker square of a 2-by-2 matrix to obtain similar sums for the Fibonacci numbers. For the function f they took the exponential function, and remarked that analogous results could be obtained by using the circular and hyperbolic functions. Identities (3.4), (3.5), (3.8), (3.10), and (3.11), respectively, generalize identities (12)-(16) of [5].

4. APPLICATIONS

We now specialize (3.4) and (3.11) to the Chebyshev polynomials to obtain some attractive sums involving the squares of the sine and cosine functions.

Let $\{T_n(t)\}_{n=0}^{\infty}$ and $\{S_n(t)\}_{n=0}^{\infty}$ denote the Chebyshev polynomials of the first and second kinds, respectively. Then

$$\left.\begin{aligned} S_n(t) &= \frac{\sin n\theta}{\sin\theta} \\ T_n(t) &= \cos n\theta \end{aligned}\right\}, \quad t = \cos\theta,\ n \ge 0.$$

Indeed, $\{S_n(t)\}_{n=0}^{\infty}$ and $\{2T_n(t)\}_{n=0}^{\infty}$ are the sequences $\{U_n\}_{n=0}^{\infty}$ and $\{V_n\}_{n=0}^{\infty}$, respectively, generated by (1.1), where $p = 2\cos\theta$ and $q = 1$. Thus, $\alpha = e^{i\theta}$ and $\beta = e^{-i\theta}$, which are obtained by solving $x^2 - 2\cos\theta x + 1 = 0$. Further information about the Chebyshev polynomials can be found, for example, in [1].

We use the following well-known power series, each of which has the complex plane as its domain of convergence:

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}, \tag{4.1}$$

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}, \tag{4.2}$$

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}, \tag{4.3}$$

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}. \tag{4.4}$$

Now, in (3.4), taking $U_n = \sin n\theta / \sin\theta$ and replacing f by the functions in (4.1)-(4.4), we obtain, after replacing all occurrences of $k\theta$ by ϕ,

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1} \sin^2(2n+1)\phi}{(2n+1)!} = \frac{\sin x - \sin(x\cos 2\phi)\cosh(x \sin 2\phi)}{2}, \tag{4.5}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n} \sin^2 2n\phi}{(2n)!} = \frac{\cos x - \cos(x\cos 2\phi)\cosh(x \sin 2\phi)}{2}, \tag{4.6}$$

$$\sum_{n=0}^{\infty} \frac{x^{2n+1} \sin^2(2n+1)\phi}{(2n+1)!} = \frac{\sinh x - \sinh(x\cos 2\phi)\cos(x \sin 2\phi)}{2}, \tag{4.7}$$

$$\sum_{n=0}^{\infty} \frac{x^{2n} \sin^2 2n\phi}{(2n)!} = \frac{\cosh x - \cosh(x\cos 2\phi)\cos(x \sin 2\phi)}{2}. \tag{4.8}$$

Finally, in (3.11), taking $V_n = 2\cos n\theta$ and replacing f by the functions in (4.1)-(4.4), we obtain, respectively,

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1} \cos^2(2n+1)\phi}{(2n+1)!} = \frac{\sin x + \sin(x\cos 2\phi)\cosh(x \sin 2\phi)}{2}, \tag{4.9}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n} \cos^2 2n\phi}{(2n)!} = \frac{\cos x + \cos(x\cos 2\phi)\cosh(x \sin 2\phi)}{2}, \tag{4.10}$$

$$\sum_{n=0}^{\infty} \frac{x^{2n+1} \cos^2(2n+1)\phi}{(2n+1)!} = \frac{\sinh x + \sinh(x\cos 2\phi)\cos(x \sin 2\phi)}{2}, \tag{4.11}$$

$$\sum_{n=0}^{\infty} \frac{x^{2n} \cos^2 2n\phi}{(2n)!} = \frac{\cosh x + \cosh(x\cos 2\phi)\cos(x \sin 2\phi)}{2}. \tag{4.12}$$

REFERENCES

1. M. Abramowitz & I. A. Stegun. *Handbook of Mathematical Functions*. New York: Dover, 1972.
2. R. Barakat. "The Matrix Operator e^X and the Lucas Polynomials." *Journal of Mathematics and Physics* **43** (1964):332-35.
3. R. Bellman. *Introduction to Matrix Analysis*. New York: McGraw-Hill, 1970.
4. G. Berzsenyi. Solution to Problem H-274. *The Fibonacci Quarterly* **17.1** (1979):95.

5. O. Brugia & P. Filipponi. "Functions of the Kronecker Square of the Matrix Q." In *Applications of Fibonacci Numbers* **2**:69-76. Ed. A. N. Philippou et al. Dordrecht: Kluwer, 1988.
6. P. Filipponi & A. F. Horadam. "A Matrix Approach to Certain Identities." *The Fibonacci Quarterly* **26.2** (1988):115-26.
7. F. R. Gantmacher. *The Theory of Matrices*. New York: Chelsea, 1960.
8. A. F. Horadam & P. Filipponi. "Cholesky Algorithm Matrices of Fibonacci Type and Properties of Generalized Sequences." *The Fibonacci Quarterly* **29.2** (1991):164-73.
9. E. Lucas. "Théorie des Fonctions Numériques Simplement Periodiques." *American Journal of Mathematics* **1** (1878):184-240, 289-321.
10. R. S. Melham & A. G. Shannon. "Some Infinite Series Summations Using Power Series Evaluated at a Matrix." *The Fibonacci Quarterly* **33.1** (1995):13-20.
11. S. Pethe. "On Sequences Having Third Order Recurrence Relations." In *Fibonacci Numbers and Their Applications* **1**:185-92. Ed. A. N. Philippou et al. Dordrecht: Kluwer, 1986.
12. A. G. Shannon & A. F. Horadam. "Some Properties of Third-Order Recurrence Relations." *The Fibonacci Quarterly* **10.2** (1972):135-45.
13. J. E. Walton. "Lucas Polynomials and Certain Circular Functions of Matrices." *The Fibonacci Quarterly* **14.1** (1976):83-87.

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BOOK REVIEW

Richard A. Dunlap, *The Golden Ratio and Fibonacci Numbers* (River Edge, NJ: World Scientific, 1997).

This attractive and carefully written book addresses the general reader with interest in mathematics and its application to the physical and biological sciences. In addition, it provides supplementary reading for a lower division university course in number theory or geometry and introduces basic properties of the golden ratio and Fibonacci numbers for researchers working in fields where these numbers have found applications.

An extensive collection of diagrams illustrate geometric problems in two and three dimensions, quasi-crystallography, Penrose tiling, and biological applications. Appendices list the first 100 Fibonacci and Lucas numbers, a collection of equations involving the golden ratio and generalized Fibonacci numbers, and a diverse list of references.

A new book on Fibonacci-related topics is published infrequently; this one will make a valuable addition to academic and personal libraries, and the many diagrams will knock your socks off.

Reviewed by Marjorie Bicknell-Johnson