THE GOLDEN SECTION AND NEWTON APPROXIMATION

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In this note we combine number theory (continued fraction convergents (see [1], ch. X) to the golden section) and calculus (Newton approximants to zeros (see [3], ch. 4)).

The golden section $g := \frac{\sqrt{5}-1}{2}$ satisfies $g^2 + g = 1$; for $G := g^{-1} = g + 1$, we have $G^2 = G + 1$. The even (continued fraction) convergents to g are

$$g_n := \frac{F_{2n}}{F_{2n+1}} \quad (n = 0, 1, 2, 3, \ldots).$$

The arbitrary function $H : [0, g] \to \mathbb{R}$ of class C^2 may satisfy $H(0) = 1$, $H(g) = 0$, and $H'(x) < 0$, $H''(x) > 0$ $(0 \le x < g)$. Let

$$N(x) := x - \frac{H(x)}{H'(x)};$$

then Newton approximation applies with

$$x_0 := 0, \quad x_{n+1} := N(x_n) > x_n \quad (n = 0, 1, 2, \ldots), \quad \lim_{n\to\infty} x_n = g.$$

In this note we give H explicitly such that $x_n = g_n$ $(n = 0, 1, 2, \ldots)$. For this, we look at

$$D(x) := \frac{1 - x - x^2}{2 + x} = \frac{(g - x)(G + x)}{2 + x} = \frac{(1 - Gx)(1 + gx)}{2 + x};$$

$y = D(x)$ is a hyperbola with the asymptotes $x = -2$ and $x + y = 1$. Thus, we have

$$D(-G) = D(g) = 0, \; D(-1) = 1, \; D'(-1) = 0, \; D(0) = \frac{1}{2}, \; D(x) > 0 \; (-G < x < g).$$

By $G^3 + g^3 = 2\sqrt{5}$, $G^2 - g^2 = \sqrt{5}$, we have

$$\frac{\sqrt{5}}{D(x)} = \frac{G^3}{1 - Gx} + \frac{g^3}{1 + gx}.$$

To be specific, we choose

$$H(x) := \exp\left(-\int_0^x \frac{dt}{D(t)}\right) \quad (0 \le x \le g).$$

Using log and differentiation, we find that

$$H(x) = (1 - Gx)^{G^2/\sqrt{5}}(1 + gx)^{-g^2/\sqrt{5}} \quad (0 \le x \le g)$$

and also that

$$\frac{H'(x)}{H(x)} = -\frac{1}{D(x)} \quad (0 \le x < g).$$

We observe the following:

$$H(x)>0,\quad H'(x)<0\ \ (0\le x<g),\quad H(g)=0,\quad H'(g)=0,$$

$$N(x)=x+D(x)=\frac{x+1}{x+2}=1-\frac{1}{x+2},\quad N(g)=g,\quad N'(x)=\frac{1}{(x+2)^2};$$

$y=N(x)$ is a hyperbola with the asymptotes $x=-2$, $y=1$. Thus, we have

$$N(-1)=0,\quad N(0)=\frac{1}{2}.$$

From $D(x)H'(x)+H(x)=0$, $D(x)H''(x)+N'(x)H'(x)=0$, we deduce $H''(x)>0$ $(0\le x<g)$. We also note that

$$x_0:=0,\quad x_{n+1}:=\frac{x_n+1}{x_n+2}\quad (n=0,1,2,\dots).$$

Theorem: We have $x_n=g_n$ $(n=0,1,2,\dots)$.

Proof: We know that $x_0=g_0=0$. It remains to show that

$$\frac{F_{2n+2}}{F_{2n+3}}=\frac{\frac{F_{2n}}{F_{2n+1}}+1}{\frac{F_{2n}}{F_{2n+1}}+2}\quad\text{or}\quad \frac{F_{2n+2}}{F_{2n+3}}=\frac{F_{2n}+F_{2n+1}}{F_{2n}+2F_{2n+1}}\quad (n=0,1,2,\dots);$$

but the numerators are equal and also the denominators.

For integers $a,b>0$, $c,d>0$, let $bc-ad=1$, then $(a,b)=(c,d)=1$, and

$$\frac{a}{b}<\frac{a+c}{b+d}\ \text{("mediant")}\ <\frac{c}{d}.$$

Let $a':=a+b$, $b':=a+2b>0$, $c':=c+d$, $d':=c+2d>0$, then

$$(a',b')=(a+b,a+2b)=(a+b,b)=(a,b)=1,\quad (c',d')=\cdots=1,$$

$$N\left(\frac{a}{b}\right)=\frac{a'}{b'},\quad N\left(\frac{a+c}{b+d}\right)=\frac{(a+c)+(b+d)}{(a+c)+2(b+d)}=\frac{a'+c'}{b'+d'},\quad N\left(\frac{c}{d}\right)=\frac{c'}{d'};$$

hence, N respects mediants.

I treated this topic during my visit to Johannesburg in 1985 (see [2]). I am grateful to the referee for a careful reading of the manuscript.

REFERENCES

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