ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-572 ***Proposed by Paul S. Bruckman, Berkeley, CA***

Prove the following, where $\varphi = \alpha^{-1}$:

$$\sum_{n=0}^{\infty}\{\varphi^{5n+1}/(5n+1)+\varphi^{5n+3}/(5n+2)-\varphi^{5n+4}/(5n+3)-\varphi^{5n+4}/(5n+4)\} = (\pi/25)(50-10\sqrt{5})^{1/2}.$$

H-573 ***Proposed by N. Gauthier, Royal Military College of Canada***

"By definition, a magic matrix is a square matrix whose lines, columns, and two main diagonals all add up to the same sum. Consider a 3×3 magic matrix Φ whose elements are the following combinations of the n^{th} and $(n+1)^{\text{th}}$ Fibonacci numbers:

$$\begin{array}{lll} \Phi_{11} = 3F_{n+1}+F_n; & \Phi_{12} = F_{n+1}; & \Phi_{13} = 2F_{n+1}+2F_n; \\ \Phi_{21} = F_{n+1}+2F_n; & \Phi_{22} = 2F_{n+1}+F_n; & \Phi_{23} = 3F_{n+1}; \\ \Phi_{31} = 2F_{n+1}; & \Phi_{32} = 3F_{n+1}+2F_n; & \Phi_{33} = F_{n+1}+F_n. \end{array}$$

Find a closed-form expression for Φ^m, where m is a *positive* integer, and determine all the values of m for which it too is a magic matrix."

SOLUTIONS

Geometric?

H-561 ***Proposed by N. Gauthier, Dept. of Physics, Royal Military College of Canada (Vol. 38, no. 2, May 2000)***

Let n be an integer and set $s_{n+1} = \alpha^n + \alpha^{n-1}\beta + \cdots + \alpha\beta^{n-1} + \beta^n$, where $\alpha+\beta = a$, $\alpha\beta = b$, with $a \neq 0$, $b \neq 0$ two arbitrary parameters. Then prove that:

(a) $s_p^r s_{qr+n} = \sum_{\ell=0}^{r}\binom{r}{\ell} b^{q(r-\ell)} s_q^\ell s_{p-q}^{r-\ell} s_{p\ell+n}$;

(b) $b^{pr} s_q^r s_n = \sum_{\ell=0}^{r}(-1)^\ell \binom{r}{\ell} s_p^\ell s_{q+p}^{r-\ell} s_{q\ell+pr+n}$;

(c) $s_{2p+q}^r s_{qr+n} = \sum_{\ell=0}^{r}\binom{r}{\ell} b^{(p+q)(r-\ell)} s_p^{r-\ell} s_{p+q}^\ell s_{(2p+q)\ell-pr+n}$;

where $r \geq 0$, n, $p(\neq 0)$, and $q(\neq 0, \pm p)$ are arbitrary integers.

Solution by Paul S. Bruckman, Berkeley, CA

Let the sums indicated in the right-hand members of parts (a), (b), and (c) be denoted by $\mathscr{A}=\mathscr{A}(p,q,r,n)$, $\mathscr{B}=\mathscr{B}(p,q,r,n)$, $\mathscr{C}=\mathscr{C}(p,q,r,n)$, respectively. We distinguish two possibilities: Case 1, where $a^2 \neq 4b$, i.e., $\alpha \neq \beta$; Case 2, where $a^2 = 4b$, i.e., $\alpha = \beta$. For Case 1, the Binet formula holds: $s_n = (\alpha^n - \beta^n)/(\alpha - \beta)$; for Case 2, we have: $s_n = n\alpha^{n-1}$, in either case for all integers n. We first deal with Case 1; we also make the substitution $\theta = \alpha - \beta$. The sums may be evaluated by application of the appropriate formula for s_n and the Binomial theorem. We will also employ some auxiliary results, given in the form of lemmas.

Lemma 1: $b^q s_{p-q} + \alpha^p s_q = \alpha^q s_p$; $b^q s_{p-q} + \beta^p s_q = \beta^q s_p$.

Proof:

$$b^q s_{p-q} + \alpha^p s_q = \{(\alpha\beta)^q(\alpha^{p-q} - \beta^{p-q}) + \alpha^p(\alpha^q - \beta^q)\}/\theta = \alpha^p(\alpha^q - \beta^q)/\theta = \alpha^q s_p;$$

also,

$$b^q s_{p-q} + \beta^p s_q = \{(\alpha\beta)^q(\alpha^{p-q} - \beta^{p-q}) + \beta^p(\alpha^q - \beta^q)\}/\theta = \beta^p(\alpha^q - \beta^q)/\theta = \beta^q s_p. \ \square$$

Lemma 2: $s_{p+q} - \alpha^q s_p = \beta^p s_q$; $s_{p+q} - \beta^q s_p = \alpha^p s_q$.

Proof: In Lemma 1, replace p by $p+q$, then divide by α^q (or β^q). $\square$

Lemma 3: $\beta^{p+q} s_p + \alpha^p s_{p+q} = \alpha^{p+q} s_p + \beta^p s_{p+q} = s_{2p+q}$.

Proof: Replace q by $p+q$ in Lemma 2. $\square$

We may now proceed to the proof of the problem, at least for Case 1.

$$\textit{(a)}\quad \mathscr{A} = \sum_{k=0}^{r} {}_rC_k b^{q(r-k)}(s_q)^k(s_{p-q})^{r-k}(\alpha^{pk+n} - \beta^{pk+n})/\theta$$
$$= (\alpha^n/\theta)(b^q s_{p-q} + \alpha^p s_q)^r - (\beta^n/\theta)(b^q s_{p-q} + \beta^p s_q)^r$$
$$= (\alpha^n/\theta)(\alpha^q s_p)^r - (\beta^n/\theta)(\beta^q s_p)^r \quad \text{(using Lemma 1)}$$
$$= (s_p)^r(\alpha^{qr+n} - \beta^{qr+n})/\theta = (s_p)^r s_{qr+n}. \ \square$$

$$\textit{(b)}\quad \mathscr{B} = \sum_{k=0}^{r} (-1)^k {}_rC_k (s_p)^k (s_{p+q})^{r-k}(\alpha^{qk+pr+n} - \beta^{qk+pr+n})/\theta$$
$$= (\alpha^{pr+n}/\theta)(s_{p+q} - \alpha^q s_p)^r - (\beta^{pr+n}/\theta)(s_{p+q} - \beta^q s_p)^r$$
$$= \alpha^{pr+n}/\theta)\beta^{pr}(s_q)^r - (\beta^{pr+n}/\theta)\alpha^{pr}(s_q)^r \quad \text{(using Lemma 2)}$$
$$= b^{pr}(s_q)^r(\alpha^n - \beta^n)/\theta = b^{pr}(s_q)^r s_n. \ \square$$

$$\textit{(c)}\quad \mathscr{C} = \sum_{k=0}^{r} {}_rC_k b^{(p+q)(r-k)}(s_p)^{r-k}(s_{p+q})^k(\alpha^{(2p+q)k-pr+n} - \beta^{(2p+q)k-pr+n})/\theta$$
$$= (\alpha^{-pr+n}/\theta)(b^{p+q}s_p + \alpha^{2p+q}s_{p+q})^r - (\beta^{-pr+n}/\theta)(b^{p+q}s_p + \beta^{2p+q}s_{p+q})^r$$
$$= (\alpha^{-pr+n+pr+qr}/\theta)(\beta^{p+q}s_p + \alpha^p s_{p+q})^r - (\beta^{-pr+n+pr+qr}/\theta)(\alpha^{p+q}s_p + \beta^p s_{p+q})^r;$$

hence, using Lemma 3, $\mathscr{C} = (\alpha^{qr+n}/\theta)(s_{2p+q})^r - (\beta^{qr+n}/\theta)(s_{2p+q})^r = (s_{2p+q})^r s_{qr+n}$. $\square$

It remains to prove (a), (b), and (c) for Case 2. We first show that Lemmas 1, 2, and 3 are valid for this case as well; the modified versions of these lemmas are denoted by a "prime" punctuation.

Lemma 1': $b^q s_{p-q} + \alpha^p s_q = \alpha^q s_p$.

Proof: $b^q s_{p-q} + \alpha^p s_q = (p-q)\alpha^{2q}(\alpha^{p-q-1}) + q\alpha^p(\alpha^{q-1}) = p\alpha^{p+q-1} = p\alpha^{p-1}\alpha^q = \alpha^q s_p$. □

Lemma 2': $s_{p+q} - \alpha^q s_p = \alpha^p s_q$.

Proof: Replace p by $p+q$ in Lemma 1′ and divide by α^q. □

Lemma 3': $\alpha^{p+q}s_p + \alpha^p s_{p+q} = s_{2p+q}$.

Proof: Replace q by $p+q$ in Lemma 2′.

(a)'
$$\begin{aligned}
\mathscr{A} &= \sum_{k=0}^{r} {}_rC_k\, b^{q(r-k)}(s_q)^k(s_{p-q})^{r-k}(pk+n)\alpha^{pk+n-1} \\
&= pr\sum_{k=1}^{r} {}_{r-1}C_{k-1} b^{q(r-k)}(s_q)^k(s_{p-q})^{r-k}\alpha^{pk+n-1} + n\sum_{k=0}^{r} {}_rC_k\, b^{q(r-k)}(s_q)^k(s_{p-q})^{r-k}\alpha^{pk+n-1} \\
&= pr\sum_{k=0}^{r-1} {}_{r-1}C_k\, b^{q(r-1-k)}(s_q)^{k+1}(s_{p-q})^{r-1-k}\alpha^{pk+p+n-1} + n\sum_{k=0}^{r} {}_rC_k\, b^{q(r-k)}(s_q)^k(s_{p-q})^{r-k}\alpha^{pk+n-1} \\
&= prs_q\alpha^{p+n-1}\{b^q s_{p-q} + \alpha^p s_q\}^{r-1} + n\alpha^{n-1}\{b^q s_{p-q} + \alpha^p s_q\}^r \\
&= pr\alpha^{p+n-1+q(r-1)} s_q(s_p)^{r-1} + n\alpha^{n-1+qr}(s_p)^r \quad \text{(using Lemma 1′)} \\
&= \alpha^{p+n-2+qr}(s_p)^{r-1}(pqr+np) = p\alpha^{p-1}(qr+n)\alpha^{qr+n-1}(s_p)^{r-1} = (s_p)^r s_{qr+n}. \quad \square
\end{aligned}$$

(b)'
$$\begin{aligned}
\mathscr{B} &= \sum_{k=0}^{r}(-1)^k\, {}_rC_k(s_p)^k(s_{p+q})^{r-k}(qk+pr+n)\alpha^{qk+pr+n-1} \\
&= -qrs_p\alpha^{q+pr+n-1}\sum_{k=0}^{r-1}(-1)^k\, {}_{r-1}C_k(s_p)^k(s_{p+q})^{r-1-k}\alpha^{qk} \\
&\quad + (pr+n)\alpha^{pr+n-1}\sum_{k=0}^{r}(-1)^k\, {}_rC_k(s_p)^k(s_{p+q})^{r-k}\alpha^{qk} \\
&= -qrs_p\alpha^{q+pr+n-1}(s_{p+q} - \alpha^q s_p)^{r-1} + (pr+n)\alpha^{pr+n-1}(s_{p+q} - \alpha^q s_p)^r \\
&= -qrs_p\alpha^{q+pr+n-1}(\alpha^p s_q)^{r-1} + (pr+n)\alpha^{pr+n-1}(\alpha^p s_q)^r \quad \text{(using Lemma 2′)} \\
&= \{-rs_q s_p\alpha^{pr+n} + (pr+n)s_q\alpha^{pr+n-1+p}\}(\alpha^p s_q)^{r-1} \\
&= \{-rs_p\alpha^{2pr+n-p} + (pr+n)\alpha^{2pr+n-1}\}(s_q)^r = \{-pr\alpha^{2pr+n-1} + (pr+n)\alpha^{2pr+n-1}\}(s_q)^r \\
&= n\alpha^{2pr+n-1}(s_q)^r = \alpha^{2pr}(s_q)^r s_n = b^{pr}(s_q)^r s_n. \quad \square
\end{aligned}$$

(c)':
$$\begin{aligned}
\mathscr{C} &= \sum_{k=0}^{r} {}_rC_k\, b^{(p+q)(r-k)}(s_p)^{r-k}(s_{p+q})^k\{(2p+q)k - pr + n\}\alpha^{(2p+q)k-pr+n-1} \\
&= (2p+q)r\alpha^{2p+q-pr+n-1}s_{p+q}\sum_{k=0}^{r-1} {}_{r-1}C_k\, b^{(p+q)(r-1-k)}(s_p)^{r-1-k}(s_{p+q})^k\alpha^{(2p+q)k} +
\end{aligned}$$

$$
\begin{aligned}
&+(-pr+n)\alpha^{-pr+n-1}\sum_{k=0}^{r} {}_rC_k\, b^{(p+q)(r-k)}(s_p)^{r-k}(s_{p+q})^k\alpha^{(2p+q)k}\\
&=(2p+q)r\alpha^{2p+q-pr+n-1}s_{p+q}(b^{p+q}s_p+\alpha^{2p+q}s_{p+q})^{r-1}\\
&\quad+(-pr+n)\alpha^{-pr+n-1}(b^{p+q}s_p+\alpha^{2p+q}s_{p+q})^r\\
&=(2p+q)r\alpha^{2p+q-pr+n-1}\alpha^{(p+q)(r-1)}s_{p+q}(\alpha^{p+q}s_p+\alpha^p s_{p+q})^{r-1}\\
&\quad+(-pr+n)\alpha^{-pr+n-1}\alpha^{(p+q)r}(\alpha^{p+q}s_p+\alpha^p s_{p+q})^r\\
&=(2p+q)r\alpha^{p-1+qr+n}s_{p+q}(s_{2p+q})^{r-1}+(-pr+n)\alpha^{qr+n-1}(s_{2p+q})^r\\
&=r\alpha^{p-1+qr+n-2p-q+1}s_{p+q}(s_{2p+q})^r+(-pr+n)\alpha^{qr+n-1}(s_{2p+q})^r\\
&=r(p+q)\alpha^{-p+qr+n-q+p+q-1}(s_{2p+q})^r+(-pr+n)\alpha^{qr+n-1}(s_{2p+q})^r\\
&=r(p+q)\alpha^{qr+n-1}(s_{2p+q})^r+(-pr+n)\alpha^{qr+n-1}(s_{2p+q})^r\\
&=(qr+n)\alpha^{qr+n-1}(s_{2p+q})^r=(s_{2p+q})^r s_{qr+n}.\ \square
\end{aligned}
$$

This completes the proof of the problem.

Also solved by H. Kwong, H.-J. Seiffert, and the proposer.

Greatest Problem

H-562 ***Proposed by H.-J. Seiffert, Berlin, Germany (Vol. 38, no. 2, May 2000)-corrected***

Show that, for all nonnegative integers n,

$$F_{2n+1}=4^n-5\sum_{k=0}^{\left[\frac{n-2}{5}\right]}\binom{2n+1}{n-5k-2},$$

where $[\cdot]$ denotes the greatest integer function.

Solution by the proposer

Define the Fibonacci polynomials by $F_0(x)=0$, $F_1(x)=1$, and $F_{n+2}(x)=xF_{n+1}(x)+F_n(x)$, for $n\in N_0$ (natural numbers).

Let $A_n:=i^{n-1}F_n(i\alpha)$, $n\in N_0$, where $i=\sqrt{(-1)}$. Writing $n=5m+r$, where $m\in N_0$ and $r\in\{0,1,2,3,4\}$, a simple induction proof on m yields

$$A_n=\begin{cases}0 & \text{if } n\equiv 0 \pmod 5,\\ 1 & \text{if } n\equiv 1 \pmod 5,\\ -\alpha & \text{if } n\equiv 2 \pmod 5,\\ \alpha & \text{if } n\equiv 3 \pmod 5,\\ -1 & \text{if } n\equiv 4 \pmod 5.\end{cases}\tag{1}$$

From H-518 [identity (8)], we know that, for all complex numbers x and all nonnegative integers n,

$$\sum_{j=0}^{n}(-1)^{j-1}\binom{2n}{n-j}F_j(x)^2=\frac{4^n-(-x^2)^n}{4+x^2}.$$

With $x = i\alpha$, this gives

$$\sum_{j=0}^{n} \binom{2n}{n-j} A_n^2 = \frac{4^n - \alpha^{2n}}{4-\alpha^2}. \tag{2}$$

Since $1/(4-\alpha^2) = \alpha/\sqrt{5}$, we have

$$\frac{4^n - \alpha^{2n}}{4-\alpha^2} = \frac{1}{\sqrt{5}}(4^n\alpha - \alpha^{2n+1}),$$

or, by $\alpha^r = (L_r + \sqrt{5}F_r)/2$, $r \in Z$,

$$\frac{4^n - \alpha^{2n}}{4-\alpha^2} = \frac{\sqrt{5}}{10}(4^n - L_{2n+1}) + \frac{1}{2}(4^n - F_{2n+1}). \tag{3}$$

On the other hand, from (1), it follows that

$$\sum_{j=0}^{n} \binom{2n}{n-j} A_j^2 = \sum_{k=0}^{[\frac{n-1}{5}]} \binom{2n}{n-5k-1} + \alpha^2 \sum_{k=0}^{[\frac{n-2}{5}]} \binom{2n}{n-5k-2}$$
$$+ \alpha^2 \sum_{k=0}^{[\frac{n-3}{5}]} \binom{2n}{n-5k-3} + \sum_{k=0}^{[\frac{n-4}{5}]} \binom{2n}{n-5k-4}.$$

Using $\alpha^2 = (3+\sqrt{5})/2$, we then obtain

$$\sum_{j=0}^{n} \binom{2n}{n-j} A_j^2 = \frac{\sqrt{5}}{2}\left(\sum_{k=0}^{[\frac{n-2}{5}]} \binom{2n}{n-5k-2} + \sum_{k=0}^{[\frac{n-3}{5}]} \binom{2n}{n-5k-3} \right) + q_n,$$

where q_n is a rational number. Since $\sqrt{5}$ is irrational, (2), (3), and the latter equation imply that

$$\sum_{k=0}^{[\frac{n-2}{5}]} \binom{2n}{n-5k-2} + \sum_{k=0}^{[\frac{n-3}{5}]} \binom{2n}{n-5k-3} = \frac{1}{5}(4^n - L_{2n+1}).$$

This proves the desired identity, because

$$\binom{2n}{n-5k-2} + \binom{2n}{n-5k-3} = \binom{2n+1}{n-5k-2}, \quad 0 \le k \le \left[\frac{n-2}{5}\right],$$

where we set $\binom{2n}{j} = 0$ for $j < 0$.

Also solved by H. Kwong and P. Bruckman.

A Stirling Problem

H-563 ***Proposed by N. Gauthier, Dept. of Physics, Royal Military College of Canada (Vol. 38, no. 2, May 2000)***

Let $m > 0$, $n \ge 0$, $p \ne 0$, $q \ne -p, 0$, and s be integers and, for $1 \le k \le n$, let $(n)_k := n(n-1) \ldots (n-k+1)$ and $S_m^{(k)}$ be a Stirling number of the second kind.

Prove the following identity for Fibonacci numbers:

$$\sum_{r=0}^{n} (-1)^r \binom{n}{r} r^m [F_p / F_{p+q}]^r F_{qr+s} = (-1)^{np} [F_q / F_{p+q}]^n \sum_{k=1}^{m} (-1)^{(p+1)k} (n)_k S_m^{(k)} [F_p / F_q]^k F_{(p+q)k-np+s}.$$

Solution by the proposer

For x an arbitrary variable, consider the binomial expansion identity

$$\sum_{r=0}^{n}\binom{n}{r}x^r = (1-x)^n,$$

and apply the operator $D := x\frac{d}{dx}$ to it m times $(m>0)$ to get

$$\sum_{r=0}^{n} r^m \binom{n}{r}x^r = D^m(1+x)^n.$$

By a well-known result,

$$D^m = \sum_{k=1}^{m} S_m^{(k)} x^k \frac{d^k}{dx^k},$$

where $\{S_m^k : 1 \le k \le m,\ 1 \le m\}$ are the Stirling numbers of the second kind. Consequently,

$$\sum_{r=0}^{n} r^m \binom{n}{r}x^r = \sum_{k=1}^{m} S_m^{(k)} x^k \frac{d^k}{dx^k}(1+x)^n = \sum_{k=1}^{m} S_m^{(k)} x^k (n)_k (1+x)^{n-k}. \qquad (*)$$

Next, for integers $p\ (\neq 0)$, $q\ (\neq 0, -p)$, solve the following for u and w:

$$1+u\alpha^q = w\alpha^{-p};\ \ 1+u\beta^q = w\beta^{-p}.$$

One readily finds

$$u = -[F_p / F_{p+q}];\ \ w = (-1)^p [F_q / F_{p+q}]. \qquad (**)$$

Inserting $x = u\alpha^q$, $1+x = w\alpha^{-p}$ in $(*)$ and multipying the resulting equation by a^s then gives

$$\sum_{r=0}^{n} r^m \binom{n}{r} u^r \alpha^{qr+s} = \sum_{k=1}^{m} S_m^{(k)} (n)_k u^k \alpha^{qk+s} w^{n-k} \alpha^{kp-np}.$$

Finally, replace α and β in this result, subtract from the above, and divide by $\sqrt{5}$ to get

$$\sum_{r=0}^{n} r^m \binom{n}{r} u^r F_{qr+s} = \sum_{k=1}^{m} S_m^{(k)} (n)_k u^k w^{n-k} F_{(p+q)k-np+s}.$$

Inserting the values for u and w from $(**)$ then establishes the result claimed in the problem statement.

The case $m=0$ is readily dealt with, and one gets

$$\sum_{r=0}^{n} (-1)^r \binom{n}{r} \left(\frac{F_p}{F_{p+q}}\right)^r F_{qr+s} = (-1)^{pn} \left(\frac{F_q}{F_{p+q}}\right)^n F_{-pn+s}.$$

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