ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Russ Euler and Jawad Sadek

Please submit all new problem proposals and corresponding solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by November 15, 2001. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \; F_0 = 0, \; F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n, \; L_0 = 2, \; L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-916 ***Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria***

Determine the value of $\prod_{k=0}^{n}(L_{2\cdot 3^k} - 1)$.

B-917 ***Proposed by José Luis Díaz, Universitat Politècnica de Catalunya, Terrassa, Spain***

Find the following sums:

(a) $\sum_{n\geq 0} \frac{1+L_{n+1}}{L_n L_{n+2}}$, *(b)* $\sum_{n\geq 0} \frac{L_{n-1}L_{n+2}}{L_n^2 L_{n+1}^2}$,

where L_k is the k^{th} Lucas number.

B-918 ***Proposed by M. N. Deshpande, Institute of Science, Nagpur, India***

Let i and j be positive integers such that $1 \leq j \leq i$. Let

$$T(i, j) = F_j F_{i-j+1} + F_j F_{i-j+2} + F_{j+1} F_{i-j+1}.$$

Determine whether or not

$$\underset{j}{\text{maximum}}\; T(i, j) - \underset{j}{\text{minimum}}\; T(i, j)$$

is divisible by 2 for all $i \geq 3$.

B-919 ***Proposed by Richard André-Jeannin, Cosnes et Romain, France***

Solve the equation $L_nF_{n+1} = p^m(p^m - 1)$, where m and n are natural numbers and p is a prime number.

B-920 ***Proposed by N. Gauthier, Royal Military College of Canada***

Prove that

$$\sum_{n=1}^{\infty} \sin\left(\frac{p\pi}{2}\cdot\frac{F_{n-1}}{F_nF_{n+1}}\right)\cos\left(\frac{p\pi}{2}\cdot\frac{F_{n+2}}{F_nF_{n+1}}\right) = 0$$

for p an arbitrary integer.

SOLUTIONS

It's A Toss

B-899 ***Proposed by David M. Bloom, Brooklyn College of CUNY, Brooklyn, NY (Vol. 38, no. 2, May 2000)***

In a sequence of coin tosses, a *single* is a term (H or T) that is not the same as any adjacent term. For example, in the sequence HHTHHHTH, the singles are the terms in positions 3, 7, and 8. Let $S(n, r)$ be the number of sequences of n coin tosses that contain exactly r singles. If $n \ge 0$ and p is a prime, find the value modulo p of $\frac{1}{2}S(n+p-1,\ p-1)$.

Solution by the proposer

Answer: $\frac{1}{2}S(n+p-1, p-1) \equiv \begin{cases} 0, & \text{if } p \nmid n, \\ F_{m-1}, & \text{if } n = mp, \end{cases} \pmod{p}.$

Proof (with a few omissions): For $n = 0$, both sides equal 1; for $n = 1$, both sides are zero. Assuming $n \ge 2$, the n nonsingles in the sequence must appear in k blocks of lengths ≥ 2, where $1 \le k \le n/2$. For fixed k, the number of ways to choose the corresponding sequence of k block lengths each ≥ 2 (with sum n) equals the number of ways to partition a string of length $n-k$ into k *nonempty* blocks, namely, $\binom{n-k-1}{k-1}$. Once the k block lengths are given, the sequence of tosses is determined by (1) our choice of which $p-1$ of the $k+p-1$ blocks (including the singles) are singles, and (2) whether the sequence begins with H or T. Hence,

$$\frac{1}{2}S(n+p-1, p-1) = \sum_{1\le k\le n/2} \binom{k+p-1}{p-1}\binom{n-k-1}{k-1}. \tag{1}$$

In what follows, it will be convenient to use the notation $[a, b]$ for the product $a(a-1)(a-2)\cdots(b)$ when $a \ge b$.

If $p \nmid k$, the factor $\binom{k+p-1}{p-1} = \frac{[k+p-1,\, k+1]}{[p-1,\, 1]}$ in (1) is divisible by p, since p divides the numerator but not the denominator. If $p \mid k$ but $p \nmid n$ (say $k = jp$, $n-k-1 = qp-r$ where $2 \le r \le p$), then

$$\binom{n-k-1}{k-1} = \binom{qp-r}{jp-1} = \frac{[(q-j)p-0, (q-j)p-(r-2)]}{[qp-1, qp-(r-1)]}\binom{qp-1}{jp-1}$$
$$= \frac{(\text{multiple of } p)}{(\text{nonmultiple of } p)}\cdot(\text{integer}) = (\text{multiple of } p).$$

Our conclusion follows in the case $p \nmid n$; for the case $n = mp$, we need consider only those summands in (1) for which $k = jp$. Thus, we must show that

$$\sum_{1\le j\le m/2}\binom{jp+p-1}{p-1}\binom{(m-j)p-1}{jp-1}\equiv F_{m-1}\pmod{p}. \tag{2}$$

Both sides of (2) are zero when $m=1$, so assume $m\ge 2$. First, note that $\binom{jp+p-1}{p-1}=\frac{[jp+p-1,\,jp+1]}{[p-1,\,1]}\equiv 1$ (mod p) since corresponding factors of numerator and denominator are congruent (mod p) and are all nonzero (mod p). A more complicated argument of similar character (omitted here) shows that $\binom{(m-j)p-1}{jp-1}\equiv\binom{m-j-1}{j-1}$ (mod p). Hence, (2) reduces to

$$\binom{m-2}{0}+\binom{m-3}{1}+\cdots\equiv F_{m-1}\pmod{p}. \tag{3}$$

But the left side of (3) in fact *equals* F_{m-1} (well known and easily shown by induction). Q.E.D.

Also solved by Paul S. Bruckman and Kathleen Lewis.

Always Rational

B-900 *Proposed by Richard André-Jeannin, Cosnes et Romain, France (Vol. 38, no. 4, August 2000)*

Show that $\tan(2n\arctan(\alpha))$ is a rational number for every $n\ge 0$.

Solution by the proposer

We will use the well-known relation

$$\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\cdot\tan y},$$

where the values x, y, $x+y$ are different from an odd multiple of $\frac{\pi}{2}$. Further, it is clear that $\tan(\arctan\alpha)=\alpha$.

Now we will prove the given statement by induction. Denote $D_n=\tan(2n\arctan\alpha)$ for every $n\ge 0$.

It is easy to see that $D_0=0$ and

$$D_1=\tan(2\arctan\alpha)=\frac{2\tan(\arctan\alpha)}{1-\tan^2(\arctan\alpha)}=\frac{2\alpha}{1-\alpha^2}=\frac{2\alpha}{1-(1+\alpha)}=-2.$$

Suppose that D_n is a rational number, then we will show that D_{n+1} is also a rational number. We can write

$$\begin{aligned}D_{n+1}&=\tan(2(n+1)\arctan\alpha)=\tan(2n\arctan\alpha+2\arctan\alpha)\\&=\frac{\tan(2n\arctan\alpha)+\tan(2\arctan\alpha)}{1-\tan(2n\arctan\alpha)\cdot\tan(2\arctan\alpha)}=\frac{D_n+D_1}{1-D_n\cdot D_1}=\frac{D_n-2}{1+2D_n}.\end{aligned}$$

But this means that D_{n+1} is also a rational number and the proof is finished. Moreover, we have found the recurrence for D_n.

It still remains to show that, for all natural n, the condition $D_n\ne-\frac{1}{2}$ holds. If n is even, then $n=2k$, where k is a natural number. Then we have the relation

$$D_{2k}=\frac{2D_k}{1-D_k^2}=-\frac{1}{2},$$

which can be rewritten as a quadratic equation $D_k^2-4D_k-1=0$. But its roots are irrational numbers $D_k=2+\sqrt{5}$ and $D_k=2-\sqrt{5}$. This contradicts the fact that D_k must be a rational number.

If n is odd, then $n=4k+1$ or $n=4k+3$, where k is a natural number. Similarly, we can show that, if $D_{4k+1}=-\frac{1}{2}$ or $D_{4k+3}=-\frac{1}{2}$, then D_k is an irrational number.

Also solved by Paul S. Bruckman, M. Deshpande, L. A. G. Dresel, Steve Edwards, Walther Janous, Harris Kwong, Kee-Wai Lau, Reiner Martin, Don Redmond, H.-J. Seiffert, and Indulis Strazdins.

Back to Euler

B-901 ***Proposed by Richard André-Jeannin, Cosnes et Romain, France (Vol. 38, no. 4, August 2000)***

Let A_n be the sequence defined by $A_0=1$, $A_1=0$, $A_n=(n-1)(A_{n-1}+A_{n-2})$ for $n\geq 2$. Find

$$\lim_{n\to+\infty}\frac{A_n}{n!}.$$

Solution by Paul S. Bruckman, Berkeley, CA

This problem is an old one, and occurs in the study of the number of *derangements* of n objects. We may express this in the following manner: In how many ways can the ordered set of integers $\{1,2,3,\ldots,n\}$ be permuted so that, in the new arrangement, none of these integers lies in its natural order? The answer turns out to be A_n, and an interesting explicit expression for A_n may be derived, which follows:

$$A_n=n!\sum_{k=0}^{n}(-1)^k/k! \tag{1}$$

In the older literature, this expression is denoted $n!!$. Here, we simply verify that the expression for A_n in (1) satisfies the conditions of the problem. By a change in variable, the given recurrence relation becomes $A_{n+1}=n(A_n+A_{n-1})$. Note that the initial conditions are satisfied by (1) for $n=0$ and 1. Now, assuming that (1) is true for n and $n-1$, we have

$$n(A_n+A_{n-1})=n\cdot n!\sum_{k=0}^{n}(-1)^k/k!+n\cdot(n-1)!\sum_{k=0}^{n-1}(-1)^k/k!=(n+1)\cdot n!\sum_{k=0}^{n}(-1)^k/k!-n!(-1)^n/n!$$

$$=(n+1)!\sum_{k=0}^{n}(-1)^k/k!+(n+1)!(-1)^{n+1}/(n+1)!=(n+1)!\sum_{k=0}^{n+1}(-1)^k/k!=A_{n+1}.$$

Applying induction establishes (1). We then see that

$$\lim_{n\to\infty}A_n/n!=\sum_{k=0}^{\infty}(-1)^k/k!=e^{-1}.$$

The featured solution sums up all comments and solutions of the other solvers. Several solvers gave references as to where equality (1) can be found. Harris Kwong cited Combinatorial Mathematics *by H. J. Ryser, Kee-Wai Lau listed* The Encyclopedia of Integer Sequences, *by N. J. A. Sloane and Simon Plouffe, H.-J. Seiffert included* Discrete and Combinatorial Mathematics, *2nd Edition, Exercise 9 on page 402, and Indulis Strazdins (to whom we owe the title of this problem) mentioned* Introduction to Combinatorial Analysis *by J. Riordan, Chapter 3, and Exercise B-853. He also mentioned that the recursion can be traced back to Euler.*

Also solved by Michael S. Becker, M. Deshpande, L. A. G. Dresel, Walther Janous, Harris Kwong, Kee-Wai Lau, Reiner Martin, Helmut Prodinger, H.-J. Seiffert, Indulis Strazdins, and the proposer.

A Pell Polynomials Identity

B-902 ***Proposed by H.-J. Seiffert, Berlin, Germany***
(Vol. 38, no. 4, August 2000)

The Pell polynomials are defined by $P_0(x)=0$, $P_1(x)=1$, and $P_n(x)=2xP_{n-1}(x)+P_{n-2}(x)$ for $n\geq 2$. Show that, for all nonzero real numbers x and all positive integers n,

$$\sum_{k=1}^{n}\binom{n}{k}(1-x)^{n-k}P_k(x)=x^{n-1}P_n(1/x).$$

Solution by Reiner Martin, New York, NY

It is well known (and can also easily be verified using induction) that

$$2\sqrt{x^2+1}\cdot P_n(x)=\left(x+\sqrt{x^2+1}\right)^n-\left(x-\sqrt{x^2+1}\right)^n.$$

Thus,

$$\begin{aligned}&2\sqrt{x^2+1}\cdot\sum_{k=1}^{n}\binom{n}{k}(1-x)^{n-k}P_k(x)\\&=\sum_{k=1}^{n}\binom{n}{k}(1-x)^{n-k}\left(x+\sqrt{x^2+1}\right)^k-\sum_{k=1}^{n}\binom{n}{k}(1-x)^{n-k}\left(x-\sqrt{x^2+1}\right)^k\\&=\left(1+\sqrt{x^2+1}\right)^n-\left(1-\sqrt{x^2+1}\right)^n=2\sqrt{x^2+1}\cdot x^{n-1}P_n(1/x).\end{aligned}$$

Almost all other solvers used a similar method to prove the equality.

Also solved by Richard André-Jeannin, Paul S. Bruckman, Johan Cigler, L. A. G. Dresel, Walther Janous, Harris Kwong, Kee-Wai Lau, Helmut Prodinger, and the proposer.

An Old Generation Function

B-903 ***Proposed by the editor***
(Vol. 38, no. 4, August 2000)

Find a closed form for $\sum_{n=0}^{\infty}F_n^2x^n$.

Solution by Walther Janous, Innsbruck, Austria

Because of $F_n^2=\frac{1}{5}(\alpha^n-\beta^n)^2=\frac{1}{5}(\alpha^{2n}+\beta^{2n}-2(-1)^n)$, we infer

$$\begin{aligned}\sum_{n=0}^{\infty}F_n^2x^n&=\frac{1}{5}\sum_{n=0}^{\infty}((\alpha^2x)^n+(\beta^2x)^n-2(-x)^n)\\&=\frac{1}{5}\left(\frac{1}{1-\alpha^2x}+\frac{1}{1-\beta^2x}-\frac{2}{1+x}\right)=\frac{1}{5}\cdot\frac{-80x(x-1)}{16(x+1)(x^2-3x+1)}\\&=\frac{x(1-x)}{(x+1)(x^2-3x+1)}\quad\left[=-\frac{x^2-x}{x^3-2x^2-2x+1}\right].\end{aligned}$$

(The domain of convergence of this expression is $\{x/|x|<\frac{1}{\alpha^2}\}$.)

The problem is well known, as some solvers pointed out. Harris Kwong mentioned some references where the problem had been generalized to any power of F_n [see J. Riordan, "Generating Functions for Powers of Fibonacci Numbers," *Duke Math. J.* **29** (1962):5-12] *and even extended* *to greater generality* [see L. Carlitz, "Generating Functions for Powers of Certain Sequences of Numbers," *Duke Math J.* **29** (1962):521-38.] *Some solvers used Maple to produce solutions for up to the 10th power of* F_n. *Richard André-Jeannin mentioned an article by Verner E. Hoggatt, Jr.* ["Some Special Fibonacci and Lucas Generating Functions," *The Fibonacci Quarterly* **9.2** (1971):121-23] *and H.-J. Seiffert commented that the answer to this problem was given in the solution to B-452* [*The Fibonacci Quarterly* **20.3** (1982):280-81]. *Pentti Haukkanen also cited three additional references.*

Also solved by Richard André-Jeannin, Paul S. Bruckman, Charles K. Cook, M. Deshpande, L. A. G. Dresel, Pentti Haukkanen, Harris Kwong, Joe Lewis, Reiner Martin, Jalis Morrison, Helmut Prodinger, Maitland Rose, Don Redmond, H.-J. Seiffert, Pantelimon Stănică, Indulis Strazdins, and the proposer.

A Fibonacci-Lucas Equality

B-904 ***Proposed by Richard André-Jeannin, Cosnes et Romain, France (Vol. 38, no. 4, August 2000)***

Find the positive integers n and m such that $F_n = L_m$.

Solution by Harris Kwong, Fredonia, NY

For $m \geq 3$, we have $F_{m+1} < L_m < F_{m+2}$, because

$$L_m = F_{m+1} + F_{m-1} \geq F_{m+1} + F_2 > F_{m+1} \quad \text{and} \quad L_m = F_{m+1} + F_{m-1} < F_{m+1} + F_m = F_{m+2}.$$

Therefore, $F_n = L_m$ only when $m < 3$. The only solutions are $(n, m) = (1, 1), (2, 1), (4, 2)$.

Also solved by Richard André-Jeannin, Brian D. Beasley, Paul S. Bruckman, L. A. G. Dresel, Walther Janous, Lake Superior Problem Solving Group, Reiner Martin, Ibrahim Al-Pasari, H.-J. Seiffert, Indulis Strazdins, and the proposer.

A Three-Term Sum

B-905 ***Proposed by José Luis Díaz, Universitat Politècnica de Catalunya, Terrassa, Spain (Vol. 38, no. 4, August 2000)***

Let n be a positive integer greater than or equal to 2. Determine

$$\frac{(F_n^2+1)F_{n+1}F_{n+2}}{(F_{n+1}-F_n)(F_{n+2}-F_n)} + \frac{F_n(F_{n+1}^2+1)F_{n+2}}{(F_n-F_{n+1})(F_{n+2}-F_{n+1})} + \frac{F_nF_{n+1}(F_{n+2}^2+1)}{(F_n-F_{n+2})(F_{n+1}-F_{n+2})}.$$

Solution by Maitland A. Rose, Sumter, SC

If we replace $(F_{n+2} - F_n)$ by F_{n+1}, $(F_{n+2} - F_{n+1})$ by F_n, and simplify, the expression becomes

$$\frac{(F_n^2+1)F_{n+2}}{(F_{n+1}-F_n)} - \frac{(F_{n+1}^2+1)F_{n+2}}{(F_{n+1}-F_n)} + (F_{n+2}^2+1) = -\frac{(F_{n+1}^2-F_n^2)F_{n+2}}{(F_{n+1}-F_n)} + (F_{n+2}^2+1)$$

$$= -(F_{n+1}+F_n)F_{n+2} + (F_{n+2}^2+1) = -F_{n+2}^2 + F_{n+2}^2 + 1 = 1.$$

Also solved by Richard André-Jeannin, Brian D. Beasley, Scott H. Brown, Paul S. Bruckman, Julie Clark, Charles K. Cook, M. Deshpande, L. A. G. Dresel, Walther Janous, Harris Kwong, Carl Libis, Reiner Martin, H.-J. Seiffert, James Sellers, Pantelimon Stănică, Indulis Strazdins, and the proposer.

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