ON IRRATIONAL VALUED SERIES INVOLVING GENERALIZED FIBONACCI NUMBERS II

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1. INTRODUCTION

In the predecessor to this paper (see [7]) a family of rational termed series having irrational sums was constructed. These series whose terms, for a fixed $k \in N \setminus \{0\}$, were formed from the reciprocal of the factorial-like product of generalized Fibonacci numbers $U_k U_{k+1} \dots U_{k+n}$, in addition exhibited irrational limits when summed over arbitrary infinite subsequences of N, by replacing n with a strictly monotone increasing function $f : N \to N$. Owing to this factorial-like form, the argument employed in [7] was closely modeled on that of Euler's for establishing the irrationality of e. However, as a consequence of the approach taken, one needed to restrict attention to those sequences $\{U_n\}$, generated with respect to the relatively prime pair (P, Q) with $|Q| = 1$ and $|P| > 1$. In view of these results it was later conjectured in [7] whether other irrational valued series could be constructed having terms formed from the reciprocal of such products as $U_{f(n)} \dots U_{f(n)+k}$, where again $f : N \to N \setminus \{0\}$ was a strictly monotone increasing function. In this paper we shall provide evidence to support the conjecture by examining two disparate cases, namely, when $f(\cdot)$ satisfies a linear and an exponential growth condition. To help establish the result in the later case, a sufficient condition for irrationality will be derived. This condition, which is similar but slightly more restrictive than that employed in [2] and [6], will be demonstrated, for interest's sake, by an alternate proof based on the following well-known criterion for irrationality (see [8]): If there exists a $\delta > 0$ and a nonconstant infinite sequence $\{p_n / q_n\}$ of rational approximations to θ, with $(p_n, q_n) = 1$, and such that, for all n sufficiently large,

$$\left| \theta - \frac{p_n}{q_n} \right| < \frac{1}{q_n^{1+\delta}},$$

then θ is irrational. In addition, the above sufficiency condition will allow us to prove that the conjecture also holds for generalized Lucas sequences $\{V_n\}$, when $f(\cdot)$ has exponential growth. One notable feature of these results compared to those obtained in [7] is that they apply to a much wider family of sequences, namely, those which are generated with respect to the relatively prime pair (P, Q) with $|Q| > 1$ and $P > |Q+1|$. Unfortunately, in the linear case (i.e., when $f(n) = n$), we cannot achieve the same level of generality, as irrational sums can only be deduced for those series involving generalized Fibonacci numbers where $Q = 1$ and $P > 2$. This restriction is due to the fact that (for k even) the sum of the series in question is given as a linear expression in s over the rationals, where $s = \sum_{n=1}^{\infty} 1/U_n$ is at present only known to be irrational for $|Q| = 1$. In the final part of the paper, we return to the family of series first considered in [7] and, by applying the above criterion for irrationality, we extend the results there to encompass those series involving both the generalized Fibonacci and Lucas numbers where $Q \neq 0$ and $P > |Q+1|$.

2. SERIES WITH TERMS $(U_{f(n)} \dots U_{f(n)+k})^{-1}$

We will first address the case of $f(\cdot)$ satisfying an exponential growth condition. To help motivate the required sufficient condition for irrationality, let us consider the following result which was first stated (without proof) in [3] but later proved by Badea in [2].

Theorem 2.1: Let $\{a_n\}$, for $n \geq 1$, be a sequence of integers such that $a_{n+1} > a_n^2 - a_n + 1$ holds for all n. Then the sum of the series $\sum_{n=1}^{\infty} 1/a_n$ is an irrational number.

This criterion, which is based on a sufficient condition for irrationality of Brun [3], is best possible, in the sense that rational valued series can occur if the strict inequality is replaced by equality. It is clear from Theorem 2.1 that, if $a_1 > 1$ and $a_{n+1} \geq a_n^2$ for $n > 1$, then the series of reciprocals $\{1/a_n\}$ must also sum to an irrational number. Such a weaker version of the above criterion was proved indirectly by McDaniel in [6] via a descent method and later was used to establish the irrationality of $\sum_{n=1}^{\infty} 1/U_{f(n)}$, where $f: N \to N$ satisfied the inequality $f(n+1) \geq 2f(n)$. In a similar manner, by using the more restrictive condition of $\inf_{n \in N}\{a_{n+1}/a_n^2\} > 1$, we can now extend the results obtained in [6] to those series involving the reciprocal of such products as $a_n = U_{f(n)} \dots U_{f(n)+k}$. Although not essential, the advantage in using this alternate condition is that we can demonstrate irrationality via a direct proof, as opposed to the indirect arguments employed in [6]. To this end, consider now the following technical lemma.

Lemma 2.1: If $\sum_{n=1}^{\infty} 1/a_n$ is a series of rationals with $a_n \in N \setminus \{0\}$ and $\inf_{n \in N}\{a_{n+1}/a_n^2\} > 1$, then the series converges to an irrational sum.

Proof: From the above inequality, the series is clearly convergent. Denoting the sum of the series by θ, we examine the sequence of rational approximations p_n/q_n to θ generated from the n^{th} partial sums, expressed in reduced form. As $a_n > 0$, for $n \geq 1$, the terms p_n/q_n must be strictly monotone increasing and so the sequence is nonconstant. To prove the irrationality of θ it is sufficient, in view of the aforementioned criterion, to demonstrate that $|q_n\theta - p_n| = o(1)$ as $n \to \infty$. Since $(p_m, q_m) = 1$, the lowest common denominator of the m fractions in the set $\{1/a_n\}_{n=1}^{m}$ must be greater than or equal to q_m but, as $a_1 a_2 \dots a_m$ is one common denominator, we deduce that $q_m \leq a_1 \dots a_m$. Thus, again by the above inequality,

$$|q_m\theta - p_m| = \sum_{n=m+1}^{\infty} \frac{q_m}{a_n} \leq a_1 \dots a_m \sum_{n=m+1}^{\infty} \frac{1}{a_n} = b_m \sum_{n=m+1}^{\infty} \frac{a_{m+1}}{a_n}$$
$$< b_m\left(1 + \sum_{r=1}^{\infty} \frac{a_{m+r}}{a_{m+r+1}}\right) < b_m\left(1 + \sum_{r=1}^{\infty} \frac{1}{a_{m+r}}\right) < b_m(1+\theta),$$

where $b_m = (a_1 \dots a_m)/a_{m+1}$. The result will follow after showing that $b_m \to 0$ as $m \to \infty$. To this end, we consider $\log(1/b_m)$. Via assumption, there must exist a $\delta > 0$ such that $a_{n+1}/a_n^2 \geq (1+\delta)$ for all n, and so

$$\log(1/b_m) = \sum_{r=1}^{m} (\log a_{r+1} - \log a_r) + \log a_1 - \sum_{r=1}^{m} \log a_r$$
$$= \sum_{r=1}^{m} \log\left(\frac{a_{r+1}}{a_r^2}\right) + \log a_1 \geq m \log(1+\delta) \to \infty \text{ as } m \to \infty. \quad \square$$

In the case of $a_n = U_{f(n)} \dots U_{f(n)+k}$, the condition of the previous lemma can be satisfied when $f(\cdot)$ has exponential growth. We demonstrate this using the following well-known identities:

$$U_{2m} = U_m V_m, \quad U_{2m-1} = U_m^2 - QU_{m-1}^2, \quad V_m > U_m. \tag{1}$$

Theorem 2.2: Suppose $\{U_n\}$ is a generalized Fibonacci sequence generated with respect to the relatively prime pair (P, Q) with $Q \neq 0$ and $P > |Q+1|$. If, for a given $k \in N$, the function $f : N \to N \setminus \{0\}$ has the property $f(n+1) > 2f(n) + 2k$ for all $n \geq 1$, then the series $\Sigma_{n=1}^{\infty} 1/a_n$ converges to an irrational sum, where $a_n = U_{f(n)} \dots U_{f(n)+k}$.

Proof: We first note that, for the prescribed values of P and Q, $\{U_n\}$ and $\{V_n\}$ are strictly monotone increasing sequences of positive integers. To demonstrate the irrationality of the series sum, it will suffice in view of Lemma 2.1 to show that $\inf_{n \in N} \{a_{n+1} / a_n^2\} > 1$. Now, since

$$\frac{a_{n+1}}{a_n^2} = \prod_{r=0}^{k} \frac{U_{f(n+1)+r}}{U_{f(n)+r}^2},$$

observe from the assumption on $f(\cdot)$ and the identities in (1) that, for $r = 0, 1, \dots, k$ and $n \in N$,

$$\frac{U_{f(n+1)+r}}{U_{f(n)+r}^2} \geq \frac{U_{2f(n)+2k+r+1}}{U_{f(n)+r}^2} \geq \frac{U_{2f(n)+2r+1}}{U_{f(n)+r}^2} = \frac{PU_{f(n)+r}V_{f(n)+r} - QU_{2f(n)+2r-1}}{U_{f(n)+r}^2}$$

$$> \frac{PU_{f(n)+r}^2 - QU_{2f(n)+2r-1}}{U_{f(n)+r}^2} = \frac{U_{f(n)+r}^2(P-Q) + Q^2 U_{f(n)+r-1}^2}{U_{f(n)+r}^2}.$$

Consequently, as $P - Q \geq 2$, one deduces from the previous inequality that $\inf_{n \in N}\{a_{n+1} / a_n^2\} > 2^{k+1} > 1$. □

Via a similar application of Lemma 2.1, one can prove the irrationality of the above series when U_n is replaced with the terms of a generalized Lucas sequence $\{V_n\}$.

Theorem 2.3: Suppose $\{V_n\}$ is a generalized Lucas sequence generated with respect to the relatively prime pair (P, Q) with $Q \neq 0$ and $P > |Q+1|$. If, for a given $k \in N$, the function $f : N \to N \setminus \{0\}$ has the property that $f(n+1) > 2f(n) + 2k$ for all $n \geq 1$, then the series $\Sigma_{n=1}^{\infty} 1/a_n$ converges to an irrational sum, where $a_n = V_{f(n)} \dots V_{f(n)+k}$.

Proof: For the prescribed values of P and Q, it is readily seen that $\lim_{n \to \infty} V_{2n+1} / V_n^2 = \alpha > 1$. Thus, for an $0 < \varepsilon < \alpha - 1$, there must exist an $N(\varepsilon) > 0$ such that $V_{2n+1} / V_n^2 > \alpha - \varepsilon > 1$, when $n > N(\varepsilon)$. Let $N' := \min\{s \in N : f(n) > N(\varepsilon) \text{ for all } n \geq s\}$ and consider the remainder of the series given here by $\Sigma_{n=N'}^{\infty} 1/a_n$. To demonstrate the irrationality of the above series, it will suffice to prove that $\inf_{n \geq N'}\{a_{n+1} / a_n^2\} > 1$. Now, for $n \geq N'$ and $r = 0, 1, \dots, k$, one clearly must have $f(n) + r > N(\varepsilon)$ and so, from the assumption on $f(\cdot)$, observe that

$$\frac{V_{f(n+1)+r}}{V_{f(n)+r}^2} \geq \frac{V_{2f(n)+2k+r+1}}{V_{f(n)+r}^2} \geq \frac{V_{2f(n)+2r+1}}{V_{f(n)+r}^2} > \alpha - \varepsilon.$$

Consequently, $\inf_{n \geq N'}\{a_{n+1} / a_n^2\} > (\alpha - \varepsilon)^{k+1} > 1$. □

Turning now to the case of $f(n)=n$, it is readily apparent that one cannot apply Lemma 2.1 to prove irrationality as $a_{n+1}/a_n^2=\prod_{r=0}^{k}(U_{n+1+r}/U_{n+r}^2)\to 0$ as $n\to\infty$ and so the infimum over the natural numbers of the associated sequence must be equal to zero. In spite of this, one can still reach the desired conclusion for the series in question by an application of two existing results within the literature. The first of these, which is due to André-Jeannin (see [1]), asserted that the series $\sum_{n=1}^{\infty}1/U_n$ sums to an irrational number when $\{U_n\}$ is generated with respect to the ordered pair (P,Q), where $|Q|=1$ and $P>2$. By then combining this with the well-known reduction formula of Carlitz for Fibonacci summations, we can write the sum of $\sum_{n=1}^{\infty}1/a_n$ (for any fixed $k\in N$) in terms of a linear expression in θ over the rationals, where θ is an irrational number to be determined. Thus, consider now the following Lemma which forms the basis of the reduction formula that shall be used directly.

Lemma 2.2: Suppose that the sequences $\{U_n\}$ and $\{V_n\}$ are generated with respect to the ordered pair $(P,1)$ with $P\neq 1,2$ and let α and β be the roots of $x^2-Px+1=0$. If we denote $\left\{{m \atop r}\right\}=(U)_m/(U)_r(U)_{m-r}$, where $(U)_m=U_1\ldots U_m$ and $(U)_0=1$, then

$$\sum_{j=0}^{2m}(-1)^j\left\{{2m \atop j}\right\}x^j=\prod_{j=1}^{m}[1-V_{2j-1}x+x^2], \tag{2}$$

$$\sum_{j=0}^{2m+1}(-1)^j\left\{{2m+1 \atop j}\right\}\alpha^{j+1}x^j=(\alpha-\alpha^{m+2}\beta^{-m}x)\prod_{j=1}^{m}[1-V_{2j-1}x+x^2], \tag{3}$$

$$\frac{1}{U_nU_{n+1}\cdots U_{n+2m}}=\frac{1}{(U)_{2m}}\sum_{j=0}^{2m}(-1)^j\left\{{2m \atop j}\right\}\frac{1}{U_{n+j}}. \tag{4}$$

The above restriction on the value of P is required in order that the Binet formula for U_n is not indeterminant. For a proof of the above identities, interested readers should consult [4].

Theorem 2.4: Suppose the sequence $\{U_n\}$ is generated with respect to the ordered pair (P,Q), where $P>2$ and $Q=1$, then the series $\sum_{n=1}^{\infty}1/a_n$ sums to an irrational number where $a_n=U_n\ldots U_{n+k}$ and $k\in N$.

Proof: For the prescribed values of P and Q, all series under consideration are clearly convergent. Addressing the case in which k is even, observe from Lemma 2.2 when $x=1$ that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{a_n}&=\frac{1}{(U)_{2m}}\sum_{j=0}^{2m}(-1)^j\left\{{2m \atop j}\right\}\sum_{n=1}^{\infty}\frac{1}{U_{n+j}}\\&=\frac{1}{(U)_{2m}}\sum_{j=0}^{2m}(-1)^j\left\{{2m \atop j}\right\}\sum_{n=1}^{\infty}\frac{1}{U_n}-\frac{1}{(U)_{2m}}\sum_{j=1}^{2m}(-1)^j\left\{{2m \atop j}\right\}\sum_{n=1}^{j}\frac{1}{U_n}\\&=\frac{1}{(U)_{2m}}\sum_{j=1}^{m}[2-V_{2j-1}]\sum_{n=1}^{\infty}\frac{1}{U_n}-\frac{1}{(U)_{2m}}\sum_{j=1}^{2m}(-1)^j\left\{{2m \atop j}\right\}\sum_{n=1}^{j}\frac{1}{U_n}.\end{aligned}$$

Consequently, the sum of the series in question is of the form $a\theta+b$, where $a,b\in Q$ and θ is an irrational number. However, as $\{V_n\}$ is a monotone increasing sequence and $V_1=P>2$, one must have $a\neq 0$. Hence, the result is established for k even. Suppose now that $k=2m+1$, then as in [4] we multiply (4) by $1/U_{n+2m+1}$ and upon summing we find

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{a_n}&=\frac{1}{(U)_{2m}}\sum_{j=0}^{2m}(-1)^j\left\{{2m \atop j}\right\}\sum_{n=1}^{\infty}\frac{1}{U_{n+j}U_{n+2m+1}}\\&=\frac{1}{(U)_{2m}}\sum_{j=0}^{2m}(-1)^j\left\{{2m \atop j}\right\}\sum_{n=1}^{j}\frac{1}{U_nU_{n+2m-j+1}}-\frac{1}{(U)_{2m}}\sum_{j=1}^{2m}(-1)^j\left\{{2m \atop j}\right\}\sum_{n=1}^{\infty}\frac{1}{U_nU_{n+2m-j+1}}.\end{aligned}\tag{5}$$

Now, dividing both sides of the standard identity $U_{n+r}U_{n-1}-U_nU_{n+r-1}=-(\alpha\beta)^{n-1}U_r$ by the term U_nU_{n+r}, with $r=2m-j+1$, and summing to N terms, where $N>r$, observe that

$$\begin{aligned}U_{2m-j+1}\sum_{n=1}^{N}\frac{1}{U_nU_{n+2m-j+1}}&=\sum_{n=1}^{N}\frac{U_{n+2m-j}}{U_{n+2m-j+1}}-\sum_{n=1}^{N}\frac{U_{n-1}}{U_n}\\&=\sum_{n=1}^{2m-j+1}\frac{U_{N+n-1}}{U_{N+n}}-\sum_{n=1}^{2m-j+1}\frac{U_{n-1}}{U_n}.\end{aligned}\tag{6}$$

By assumption, $|\alpha|>|\beta|$, and so $U_{N+n-1}/U_{N+n}\to 1/\alpha$ as $N\to\infty$. Thus, combining the limiting value of (6) with (5), we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{a_n}&=\frac{1}{(U)_{2m}}\sum_{j=0}^{2m}(-1)^j\left\{{2m \atop j}\right\}\frac{1}{U_{2m-j+1}}\left(\frac{2m-j+1}{\alpha}-\sum_{n=1}^{2m-j+1}\frac{U_{n-1}}{U_n}\right)\\&\quad-\frac{1}{(U)_{2m}}\sum_{j=1}^{2m}(-1)^j\left\{{2m \atop j}\right\}\sum_{n=1}^{j}\frac{1}{U_nU_{n+2m-j+1}}\\&=\frac{1}{\alpha}a'+b',\end{aligned}$$

where $a',b'\in Q$ and α^{-1} is an algebraic irrational. It remains only to show that the constant $a'\neq 0$. From the definition of the generalized binomial coefficient in Lemma 2.2, we see that

$$\begin{aligned}\frac{1}{(U)_{2m}}\sum_{j=0}^{2m}(-1)^j\left\{{2m \atop j}\right\}\frac{2m-j+1}{U_{2m-j+1}}&=\frac{1}{(U)_{2m+1}}\sum_{j=0}^{2m}(-1)^j\left\{{2m \atop j}\right\}\frac{U_{2m+1}}{U_{2m-j+1}}(2m-j+1)\\&=\frac{1}{(U)_{2m+1}}\sum_{j=0}^{2m+1}(-1)^j\left\{{2m+1 \atop j}\right\}(2m-j+1).\end{aligned}$$

Thus, if we denote the polynomial function in (3) by $R(x)$, then

$$a'=\frac{2m+1}{(U)_{2m+1}}\alpha^{-1}R(\alpha^{-1})-\frac{\alpha^{-2}}{(U)_{2m+1}}\frac{dR}{dx}(\alpha^{-1}).$$

However, by Lemma 2.2, $R(x)$ contains the quadratic factor $1-V_1x+x^2=(1-\beta x)(1-\alpha x)$ and so $R(\alpha^{-1})=0$. Moreover,

$$\begin{aligned}\frac{dR}{dx}(\alpha^{-1})&=(1-V_1\alpha^{-1}+\alpha^{-2})\frac{d}{dx}\left\{(\alpha-\alpha^{m+2}\beta^{-m}x)\prod_{j=2}^{m}[1-V_{2j-1}x+x^2]\right\}(\alpha^{-1})\\&\quad+(2\alpha^{-1}-V_1)(\alpha-\alpha^{m+1}\beta^{-m})\prod_{j=2}^{m}(1-V_{2j-1}\alpha^{-1}+\alpha^{-2})\\&=(2\alpha^{-1}-V_1)(\alpha-\alpha^{m+1}\beta^{-m})\prod_{j=2}^{m}(1-V_{2j-1}\alpha^{-1}+\alpha^{-2}).\end{aligned}\tag{7}$$

Now, since $\alpha \neq \beta$, it is immediate that the first two factors in (7) must be nonzero, while as the quadratic factor $1-V_{2j-1}x+x^2$ for $j=2,\dots,m$ has the roots α^{j-1}/β^j, β^{j-1}/α^j of which neither is equal to α^{-1}, we can finally conclude that $R'(\alpha^{-1})\neq 0$ and so $a'\neq 0$. □

Remark 2.1: The inequality $P>2$ in Theorem 2.4 cannot be weakened as the series in question will sum to a rational value when $(P,Q)=(2,1)$. To demonstrate this, we first consider as $U_n=n$ in the present case, the function $f(x)=(x^{n+k}(n-1)!)/(n+k)!$. Applying Taylor's theorem to $f(x)$ about the point $a=0$, observe that, for $x\geq 0$,

$$f(x)=\sum_{m=0}^{k}\frac{x^m f^{(m)}(0)}{m!}+\int_0^x\frac{(x-t)^k}{k!}t^{n-1}\,dt. \tag{8}$$

Now, as $f^m(0)=0$ for $m=0,\dots,k$, it is clear after setting $x=1$ in (8) and applying Lebesgue's Dominated Convergence Theorem that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n(n+1)\cdots(n+k)}&=\sum_{n=1}^{\infty}\int_0^1\frac{(1-t)^k}{k!}t^{n-1}\,dt\\&=\int_0^1\sum_{n=1}^{\infty}\frac{(1-t)^k}{k!}t^{n-1}\,dt=\int_0^1\frac{(1-t)^{k-1}}{k!}dt=\frac{1}{kk!}.\end{aligned}$$

It still remains an open problem as to whether the series having terms of the above form continue to exhibit irrational sums when $f(n)$ is replaced by an arbitrary strictly monotone increasing integer valued function, such as a polynomial in n over the positive integers. Such a problem may be impossible to resolve, as it is difficult in general to predict the nature of a series sum. To illustrate this difficulty, we shall show that it is possible to construct a pair of infinite series having positive rational terms asymptotic to each other, with one summing to a rational number and the other to an irrational number. Consider $\sum_{n=1}^{\infty}1/a_n$, where a_n is generated from the recurrence relation $a_{n+1}=a_n^2-a_n+1$ with $a_1=2$. Then, as $1/(a_{n+1}-1)=1/(a_n-1)-1/a_n$, one deduces that $\sum_{n=1}^{N}1/a_n=1-(a_{N+1}-1)^{-1}$ and so the series converges to 1. However, if we define $b_n=a_n-1/n$, then clearly $1/b_n\sim 1/a_n$ as $n\to\infty$ with

$$\begin{aligned}b_{n+1}&=a_n^2-a_n+1-\frac{1}{n+1}\\&=(b_n^2-b_n+1)+\frac{2b_n}{n}+\frac{1}{n^2}-\frac{1}{n}-\frac{1}{n+1}>b_n^2-b_n+1,\end{aligned}$$

where the inequality follows from the fact that $2b_n/n+1/n^2-1/n-1/(n+1)>0$, which is easily deduced via the simple inequalities $2b_n>2$ and $2>1+n/(n+1)-1/n$. Thus, via Theorem 2.1, $\sum_{n=1}^{\infty}1/b_n$ will sum to an irrational number.

3. SERIES WITH TERMS $(U_kU_{k+1}\dots U_{k+f(n)})^{-1}$

In this section we shall again apply the criterion mentioned in the Introduction to establish irrational sums for the family of series considered in [7] but now involving the larger class of sequences $\{U_n\}$, $\{V_n\}$ generated with respect to the relatively prime pair (P,Q), where $Q\neq 0$ and $P>1$. As in the previous section, it will be convenient to first demonstrate irrationality for a general family of series having terms of the form $(x_kx_{k+1}\dots x_{k+f(n)})^{-1}$, where $\{x_n\}$ is an arbitrary strictly increasing sequence of positive integers. Although a similar result was established in [5] via an indirect argument, the version proved here is far stronger in comparison because we do not need to impose the restrictive divisibility assumption that for any $m \in N \setminus \{0\}$ there exists an n such that $m | x_1 x_2 \dots x_n$. However, it should be noted that this condition, which was also used in [7], was the source for the restriction on the parameter Q that was needed to argue in a similar manner as in [5].

Theorem 3.1: Let $\{x_n\}$ be a strictly increasing sequence of positive integers and $g: N \to N \setminus \{0\}$ a strictly monotone increasing function. If, in addition, $\{b_n\}$ is a bounded sequence of nonzero integers, then $\sum_{n=1}^{\infty} b_n / a_n$ converges to an irrational number, where $a_n = x_1 x_2 \dots x_{g(n)}$.

Proof: From the assumption, it is immediate that the series in question are absolutely convergent. Denoting the sum of the series by θ, we again consider the sequence of rational approximations p_n / q_n to θ generated from the n^{th} partial sums expressed in reduced form. As p_n / q_n are clearly nonconstant, the result will follow upon showing that $|q_n \theta - p_n| = o(1)$ as $n \to \infty$. Since $(p_n, q_n) = 1$, the lowest common denominator of the m fractions $\{b_n / a_n\}_{n=1}^{m}$ must be greater than or equal to q_m, but as $x_1 x_2 \dots x_{g(m)}$ is one common denominator, we deduce that $q_m \le x_1 x_2 \dots x_{g(m)}$. Thus, if $|b_n| \le M$ for all n, then

$$|q_m \theta - p_m| = q_m \left| \sum_{r=1}^{\infty} \frac{b_{m+r}}{a_{m+r}} \right| \le M \sum_{r=1}^{\infty} \frac{a_m}{a_{m+r}} = M \sum_{r=1}^{\infty} \frac{1}{a'_r}, \tag{9}$$

where $a'_r = x_{g(m)+1} \dots x_{g(m+r)}$. Now, by the strict monotonicity of x_n, all $g(m+r) - g(m)$ terms in the definition of a'_r are greater than or equal to $x_{g(m)+1}$. Consequently, as $g(m+r) - g(m) \ge r$, one deduces $a'_r \ge x_{g(m)+1}^{r}$, and so

$$\sum_{r=1}^{\infty} \frac{1}{a'_r} \le \sum_{r=1}^{\infty} x_{g(m)+1}^{-r} = \frac{1}{x_{g(m)+1} - 1}. \tag{10}$$

Thus, by combining (9) with (10) together with the monotonicity of x_n and $g(\cdot)$, it is readily apparent that $|q_m \theta - p_m| \to 0$ as $m \to \infty$. □

Corollary 3.1: Suppose $\{U_n\}$ and $\{V_n\}$ are generated with respect to the relatively prime pair (P, Q), with $Q \ne 0$ and $P > |Q+1|$. If, in addition, $f: N \to N \setminus \{0\}$ is a strictly monotone increasing function and $\{b_n\}$ is a bounded sequence of nonzero integers, then $\sum_{n=1}^{\infty} b_n / a_n$ converges to an irrational number, where $a_n = U_k \dots U_{k+f(n)}$ or $a_n = V_k \dots V_{k+f(n)}$.

Proof: In Theorem 3.1, substitute x_n for either U_n or V_n, which are strictly monotone increasing sequences of positive integers. If $g(n) = f(n) + k$, then $\sum_{n=1}^{\infty} b_n / a_n$ sums to an irrational number. In the case in which $k > 1$, the result will follow upon multiplying the series by the product $x_1 \dots x_{k-1}$. □

To conclude, we shall prove, as in [5], a companion result to Theorem 3.1 in which a class of irrational valued alternating series were constructed. Again one can dispense with the divisibility condition that was required in [5]; however, in its place we have imposed an order condition.

Theorem 3.2: Let $\{x_n\}$ and $\{y_n\}$ be two strictly increasing sequences of positive integers such that $y_n = o(x_n)$ as $n \to \infty$ and for all n sufficiently large,

$$\frac{y_{n+1}}{x_{n+1}} < y_n < x_n. \tag{11}$$

If, in addition, $g: N \to N \setminus \{0\}$ is a strictly monotone increasing function, then $\sum_{n=1}^{\infty}(-1)^n y_n / a_n$ converges to an irrational number where $a_n = x_1 x_2 \ldots x_{g(n)}$.

Proof: Using (11) and the fact that $g(n) \geq n$, observe that, for n sufficiently large,

$$\frac{y_{n+1}}{a_{n+1}}\frac{a_n}{y_n} = \frac{y_{n+1}}{y_n}\frac{1}{x_{g(n)+1}\cdots x_{g(n+1)}} \leq \frac{y_{n+1}}{y_n x_{g(n+1)}} \leq \frac{y_{n+1}}{y_n x_{n+1}} < 1$$

and

$$0 \leq \frac{y_n}{a_n} = (x_1 \ldots x_{g(n)-1})^{-1}\frac{y_n}{x_{g(n)}} \leq (x_1 \ldots x_{g(n)-1})^{-1}\frac{y_n}{x_n} < (x_1 \ldots x_{g(n)-1})^{-1} \to 0$$

as $n \to \infty$. So, by Leibniz' criterion, the alternating series converges. Denoting the sum of the series by θ, we again consider the sequence of rational approximations p_n / q_n to θ generated from the n^{th} partial sums expressed in reduced form. As p_n / q_n are clearly nonconstant, the result will follow upon showing that $|q_n\theta - p_n| = o(1)$ as $n \to \infty$. Since $(p_n, q_n) = 1$, the lowest common denominators of the m fractions $\{(-1)^n y_n / a_n\}_{n=1}^{m}$ must be greater than or equal to q_m, but since $x_1 x_2 \ldots x_{g(m)}$ is one common denominator, we deduce that $q_m \leq a_m$. Now

$$|q_m\theta - p_m| = q_m\left|\sum_{n=m+1}^{\infty}(-1)^n\frac{y_n}{a_n}\right| = q_m\left|\sum_{r=1}^{\infty}(-1)^{r+1}\frac{y_{m+r}}{a_{m+r}}\right| \leq a_m\left|\sum_{r=1}^{\infty}(-1)^{r+1}\frac{y_{m+r}}{a_{m+r}}\right|; \tag{12}$$

furthermore, by standard bounds from the theory of alternating series, we also have that

$$0 < \frac{y_{m+1}}{a_{m+1}} - \frac{y_{m+2}}{a_{m+2}} < \sum_{r=1}^{\infty}(-1)^{r+1}\frac{y_{m+r}}{a_{m+r}} < \frac{y_{m+1}}{a_{m+1}}.$$

Thus, we can obtain, from (12), the upper bound

$$|q_m\theta - p_m| \leq a_m\frac{y_{m+1}}{a_{m+1}} = \frac{y_{m+1}}{x_{g(m)+1}\cdots x_{g(m+1)}} \leq \frac{y_{m+1}}{x_{g(m+1)}} \leq \frac{y_{m+1}}{x_{m+1}}.$$

Hence, the result is established since, by assumption, $y_{m+1} / x_{m+1} \to 0$ as $m \to \infty$. □

As an application, we can now construct the following class of irrational valued series involving generalized Fibonacci numbers and the Euler totient function.

Corollary 3.2: Suppose $\{U_n\}$ is generated with respect to the relatively prime pair (P, Q) with $Q < 0$ and $P > 0$, then

$$\sum_{n=1}^{\infty}(-1)^n\frac{\varphi(n)}{U_1 U_2 \ldots U_n},$$

where $\varphi(n)$ is Euler's totient function, will converge to an irrational number.

Proof: In Theorem 3.2, substitute x_n for U_n, which is a strictly monotone increasing sequence of positive integers. If, in addition, we set $g(n) = n$ and $y_n = \varphi(n)$, then the irrationality of the series sum will follow if the inequality in (11) holds for n large and $\varphi(n) = o(U_n)$. To this end, we first note that, for the prescribed (P, Q) values, one must have $n = o(U_n)$, and since $\varphi(n) \le n$ for all n, we deduce that $0 < \varphi(n)/U_n < n/U_n \to 0$ as $n \to \infty$. Consequently, for n sufficiently large,

$$\frac{\varphi(n+1)}{U_{n+1}} < 1 \le \varphi(n) \le n < U_n,$$

as required. □

REFERENCES

1. R. André-Jeannin. "Irrrationalité de la comme des inverses de certaines suites récurrentes." *C. R. Acad. Sci. Paris* **308.1** (1989):539-41.
2. C. Badea. "A Theorem on Irrationality of Infinite Series and Applications." *Acta Arithmetica* **63.4** (1993):313-23.
3. V. Brun. "Ein Satz über Irrationalität." *Arch. for Math. og Naturvidenskab (Kristianin)* **31** (1910):3.
4. L. Carlitz. "Reduction Formula for Fibonacci Summations." *The Fibonacci Quarterly* **9.4** (1971):449-66.
5. N. J. Lord & J. Sandor. "On Some Irrational Series." *Math. Magazine* **65.1** (1992):53-55.
6. W. L. McDaniel. "The Irrationality of Certain Series Whose Terms Are Reciprocals of Lucas Sequence Terms." *The Fibonacci Quarterly* **32.4** (1994):346-51.
7. M. A. Nyblom. "On Irrational Valued Series Involving Generalized Fibonacci Numbers." *The Fibonacci Quarterly* **37.4** (1999):299-304.
8. A. B. Shidlovskii. *Transcendental Numbers, de Gruyter Studies in Mathematics 12*. Berlin, New York: Walter de Gruyter, 1989.

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