GENERALIZATIONS OF SOME IDENTITIES INVOLVING THE FIBONACCI NUMBERS

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The generalized Fibonacci and Lucas numbers are defined by

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad V_n = \alpha^n + \beta^n \tag{1}$$

where $\alpha = \frac{p+\sqrt{p^2-4q}}{2}$, $\beta = \frac{p-\sqrt{p^2-4q}}{2}$, $p > 0$, $q \neq 0$, and $p^2 - 4q > 0$. It is obvious that $\{U_n\}$ and $\{V_n\}$ are the usual Fibonacci and Lucas sequences $\{F_n\}$ and $\{L_n\}$ when $p = -q = 1$. Recently, for the Fibonacci numbers, Zhang established the following identities in [2]:

$$\sum_{a+b=n} F_a F_b = \frac{1}{5}((n-1)F_n + 2nF_{n-1}), \quad n \geq 1, \tag{2}$$

$$\sum_{a+b+c=n} F_a F_b F_c = \frac{1}{50}((5n^2 - 9n - 2)F_{n-1} + (5n^2 - 3n - 2)F_{n-2}), \quad n \geq 2, \tag{3}$$

and when $n \geq 3$,

$$\sum_{a+b+c+d=n} F_a F_b F_c F_d = \frac{1}{150}((4n^3 - 12n^2 - 4n + 12)F_{n-2} + (3n^3 - 6n^2 - 3n + 6)F_{n-3}). \tag{4}$$

In this paper, we extend the above conclusions. We establish some identities related to $\{U_n\}$ and $\{V_n\}$. The equalities (2)-(4) emerge as special cases of our results.

Consider the generating function of $\{U_{nk}\}$: $G_k(x) = \sum_{n=0}^{\infty} U_{nk} x^n$, where k is a positive integer. Clearly, by (1) and the geometric formula,

$$G_k(x) = \frac{U_k x}{1 - V_k x + q^k x^2}, \quad |x| > \alpha^k.$$

Let $F_k(x) = \frac{G_k(x)}{x}$. Then

$$F_k(x) = \sum_{n=1}^{\infty} U_{nk} x^{n-1} = \frac{U_k}{1 - V_k x + q^k x^2}, \quad |x| > \alpha^k. \tag{5}$$

For $F_k(x)$, we have the following lemma.

Lemma: If $F_k(x)$ is defined by (5), then $F_k(x)$ satisfies

$$F_k^2(x) = \frac{U_k}{V_k^2 - 4q^k}(F_k'(x)(V_k - 2q^k x) - 4q^k F_k(x)), \tag{6}$$

$$F_k^3(x) = \frac{U_k^2}{2(V_k^2 - 4q^k)^2}(F_k''(x)(V_k - 2q^k x)^2 - 14q^k F_k'(x)(V_k - 2q^k x) + 32q^{2k} F_k(x)), \tag{7}$$

and

$$F_k^4(x)=\frac{U_k^3}{6(V_k^2-4q^k)^3}(F_k'''(x)(V_k-2q^kx)^3-30q^kF_k''(x)(V_k-2q^kx)^2 \\ +228q^{2k}F_k'(x)(V_k-2q^kx)-384q^{3k}F_k(x)). \tag{8}$$

Proof: Noticing that

$$F_k'(x)=\frac{U_k(V_k-2q^kx)}{(1-V_kx+q^kx^2)^2}=\frac{(V_k-2q^kx)F_k(x)}{1-V_kx+q^kx^2},$$

we have

$$F_k'(x)(V_k-2q^kx)-4q^kF_k(x)=\frac{(V_k^2-4q^k)F_k(x)}{1-V_kx+q^kx^2}=\frac{V_k^2-4q^k}{U_k}F_k^2(x),$$

and hence (6) holds. Differentiating in (6), we get

$$2F_k(x)F_k'(x)=\frac{U_k}{V_k^2-4q^k}(F_k''(x)(V_k-2q^kx)-6q^kF_k'(x)).$$

Therefore,

$$2F_k(x)F_k'(x)(V_k-2q^kx)=\frac{U_k}{V_k^2-4q^k}(F_k''(x)(V_k-2q^kx)^2-6q^kF_k'(x)(V_k-2q^kx)).$$

Using (6), we have

$$2F_k(x)\left(\frac{V_k^2-4q^k}{U_k}F_k^2(x)+4q^kF_k(x)\right)=\frac{U_k}{V_k^2-4q^k}(F_k''(x)(V_k-2q^kx)^2-6q^kF_k'(x)(V_k-2q^kx)).$$

Using (6) again, we can prove that (7) holds. Similarly, differentiating in (6) and applying (6) and (7), we can obtain identity (8). □

From the above lemma, we have the main results of this paper.

Theorem: Suppose that k and n are positive integers. Then

$$\sum_{a+b=n}U_{ak}U_{bk}=\frac{U_k}{V_k^2-4q^k}((n-1)U_{nk}V_k-2q^knU_{(n-1)k}),\ n\geq 1, \tag{9}$$

$$\sum_{a+b+c=n}U_{ak}U_{bk}U_{ck}=\frac{U_k^2}{2(V_k^2-4q^k)^2}((n-1)(n-2)V_k^2U_{nk}-q^kV_k(4n^2-6n-4)U_{(n-1)k} \\ +(4n^2-28n+28(n-3)V_k+80)U_{(n-2)k}),\ n\geq 2, \tag{10}$$

and

$$\sum_{a+b+c+d=n}U_{ak}U_{bk}U_{ck}U_{dk}=\frac{U_k^3}{6(V_k^2-4q^k)^3}(V_k^3(n-1)(n-2)(n-3)U_{nk} \\ -6q^kV_k^2(n-2)(n-3)(n+1)U_{(n-1)k} \\ +12q^{2k}V_k(n-3)(n^2+n-1)U_{(n-2)k} \\ -8q^{3k}n(n^2-4)U_{(n-3)k}),\ n\geq 3. \tag{11}$$

Proof: To show that this theorem is valid, comparing the coefficients of x^{n-2}, x^{n-3}, and x^{n-4} on both sides of the Lemma, we have identities (9)-(11). □

Corollary. Assume that k and n are positive integers. Then

$$\sum_{a+b=n} F_{ak}F_{bk} = \frac{F_k}{L_k^2 - 4(-1)^k}((n-1)F_{nk}L_k - 2(-1)^k nF_{(n-1)k}),\ n \geq 1, \tag{12}$$

$$\begin{aligned}\sum_{a+b+c=n} F_{ak}F_{bk}F_{ck} = \frac{F_k^2}{2(L_k^2 - 4(-1)^k)^2}&((n-1)(n-2)L_k^2F_{nk} - (-1)^k L_k(4n^2 - 6n - 4)F_{(n-1)k} \\ &+ (4n^2 - 28n + 28(n-3)L_k + 80)F_{(n-2)k}),\ n \geq 2,\end{aligned} \tag{13}$$

and

$$\begin{aligned}\sum_{a+b+c+d=n} F_{ak}F_{bk}F_{ck}F_{dk} = \frac{F_k^3}{6(L_k^2 - 4(-1)^k)^3}&((n-1)(n-2)(n-3)L_k^3F_{nk} \\ &- 6(-1)^k(n-2)(n-3)(n+1)L_k^2F_{(n-1)k} \\ &+ 12L_k(n-3)(n^2+n-1)F_{(n-2)k} - 8(-1)^k n(n^2-4)F_{(n-3)k}),\ n \geq 3.\end{aligned} \tag{14}$$

From the Corollary, it is very easy to obtain (2)-(4). If $k = 1$ in (14), then

$$\begin{aligned}\sum_{a+b+c+d=n} F_aF_bF_cF_d = \frac{1}{750}&((n-1)(n-2)(n-3)F_n + 6(n-2)(n-3)(n+1)F_{n-1} \\ &+ 12(n-3)(n^2+n-1)F_{n-2} + 8n(n^2-4)F_{n-3}).\end{aligned}$$

By using $F_n = F_{n-1} + F_{n-2}$ $(n \geq 2)$, we can obtain (4). Similarly, from (12), (13), and $F_n = F_{n-1} + F_{n-2}$, we have identities (2) and (3). In addition, we can work out other sums from the Corollary. For example, when $k = 2$ and $q = -1$ in (13), we have

$$\sum_{a+b+c=n} F_{2a}F_{2b}F_{2c} = \frac{1}{50}(9(n-1)(n-2)F_{2n} - 3(4n^2 - 6n - 4)F_{2n-2} + (4n^2 + 56n - 172)F_{2n-4}).$$

Applying $F_n = F_{n-1} + F_{n-2}$ $(n \geq 2)$ again and again, we get

$$\sum_{a+b+c=n} F_{2a}F_{2b}F_{2c} = \frac{1}{50}((15n^2 - 63n + 66)F_{2n-3} + (10n^2 + 20n - 124)F_{2n-4}).$$

When $p = 2$ and $q = -1$ in (10), we obtain

$$\begin{aligned}\sum_{a+b+c=n} P_{ak}P_{bk}P_{ck} = \frac{P_k^2}{2(Q_k^2 - 4(-1)^k)^2}&((n-1)(n-2)Q_k^2P_{nk} - (-1)^k Q_k(4n^2 - 6n - 4)P_{(n-1)k} \\ &+ (4n^2 - 28n + 28(n-3)Q_k + 80)P_{(n-2)k}),\ n \geq 2,\end{aligned}$$

where P_k and Q_k denote the k^{th} Pell and Pell-Lucas numbers (see [1]).

REFERENCES

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