AN ANALYSIS OF n-RIVEN NUMBERS

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1. INTRODUCTION

For a positive integer a and $n \geq 2$, define $s_n(a)$ to be the sum of the digits in the base n expansion of a. If s_n is applied recursively, it clearly stabilizes at some value. Let $S_n(a) = s_n^k(a)$ for all sufficiently large k.

A Niven number [3] is a positive integer a that is divisible by $s_{10}(a)$. We define a riven number (short for recursive Niven number) to be a positive integer a that is divisible by $S_{10}(a)$. As in [2], these concepts are generalized to n-Niven numbers and n-riven numbers, using the functions s_n and S_n, respectively.

In [1], Cooper and Kennedy proved that there does not exist a sequence of more than 20 consecutive Niven numbers and that this bound is optimal. Wilson [4] determined the digit sum of the smallest number initiating a maximal Niven number sequence. The author [2] proved that, for each $n \geq 2$, there does not exist a sequence of more than $2n$ consecutive n-Niven numbers and Wilson [5] proved that this bound is optimal.

This paper presents general properties of n-riven numbers and examines the maximal possible lengths of sequences of consecutive n-riven numbers. We begin with a basic lemma characterizing the value of $S_n(a)$, which leads to many general facts about n-riven numbers. In Section 3 we determine the maximal lengths of sequences of consecutive n-riven numbers. We construct examples of sequences of maximal length for each n including ones that are provably as small as possible in terms of the values of the numbers in them.

2. BASIC PROPERTIES

Lemma 1: Fix $n \geq 2$ and $a > 0$. Then $S_n(a)$ is the unique integer such that $0 < S_n(a) < n$ and $S_n(a) \equiv a \pmod{n-1}$.

Proof: Let $a = \sum_{i=0}^{r} a_i n^i$. Then $s_n(a) = \sum_{i=0}^{r} a_i$. Since $n \equiv 1 \pmod{n-1}$, $s_n(a) \equiv a \pmod{n-1}$. Hence, for all k, $s_n^k(a) \equiv a \pmod{n-1}$, and so $S_n(a) \equiv a \pmod{n-1}$. From this, the lemma easily follows.

Corollary 2: Every positive integer is 2-riven.

Proof: It follows from Lemma 1 that, for every a, $S_2(a) = 1$.

Corollary 3: Every positive integer is 3-riven.

Proof: It follows from Lemma 1 that, for every a, $S_3(a) \equiv a \pmod 2$. So $S_3(a) = 1$ if a is odd and $S_3(a) = 2$ if a is even. Clearly, in either case, a is divisible by $S_3(a)$.

Corollary 4: For each $n \geq 2$, if a is divisible by $n-1$, then a is an n-riven number.

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Proof: If a is divisible by $n-1$, then by Lemma 1, $S_n(a)=n-1$. So a is an n-riven number.

Corollary 5: For each $n \geq 2$, there are infinitely many n-riven numbers.

3. CONSECUTIVE n-RIVEN NUMBERS

We now examine sequences of consecutive n-riven numbers. In light of Corollaries 2 and 3, we *fix a positive integer* $n \geq 4$.

Lemma 6: Let $a < b$ be numbers in a sequence of consecutive n-riven numbers. If $a \equiv b$ (mod $n-1$), then $S_n(a) | (n-1)$.

Proof: Since $a < b$ and $a \equiv b$ (mod $n-1$), $n-1 \leq b-a$. Therefore, $a+n-1 \leq b$ and so $a+n-1$ is also in the sequence of n-riven numbers. Hence, $S_n(a)|a$ and $S_n(a+n-1) | (a+n-1)$. By Lemma 1, $S_n(a+n-1)=S_n(a)$. Therefore, $S_n(a) | (a+n-1)$ and so $S_n(a) | (n-1)$.

Corollary 7: At most one number in a sequence of consecutive n-riven numbers is congruent to -1 modulo $n-1$.

Proof: Let $a < b$ be numbers in a sequence of consecutive n-riven numbers with $a \equiv b \equiv -1$ (mod $n-1$). By Lemma 6, $S_n(a)|(n-1)$. But this means that $(n-2)|(n-1)$, which is impossible for $n \geq 4$. Thus, by contradiction, no such distinct a and b can exist.

Corollary 8: There does not exist an infinitely long sequence of n-riven numbers. Equivalently, there are infinitely many numbers which are not n-riven.

Fix $m_n = \min\{k \in \mathbf{Z}^+ | k \nmid (n-1)\}$. In Theorem 9, we prove that there do not exist more than $n+m_n-1$ consecutive n-riven numbers. In Theorem 10, we prove that this bound is the best possible. Further, we find the smallest number initiating an n-riven number sequence of maximal length.

In Table 1 we present the maximal lengths of sequences of consecutive n-riven numbers for various values of n, along with the maximal sequences of minimal values.

TABLE 1. Maximal Sequences for $4 \leq n \leq 10$

n	Length	Minimal Sequence of Maximal Length
4	5	6,7,8,9,10
5	7	12,13,14,15,16,17,18
6	7	60,61,62,63,64,65,66
7	10	60,61,62,63,64,65,66,67,68,69
8	9	420,421,422,423,424,425,426,427,428
9	11	840,841,842,843,844,845,846,847,848,849,850
10	11	2520,2521,2522,2523,2524,2525,2526,2527,2528,2529,2530

Theorem 9: A sequence of consecutive n-riven numbers consists of at most $n+m_n-1$ numbers. Further, any such sequence of maximal length must start with a number congruent to zero modulo $n-1$.

Proof: Let a, $a+1$, $a+2$, ..., $a+n+m_n-2$ be a sequence of consecutive n-riven numbers and suppose $S_n(a)=k \neq n-1$.

Case 1. $1 \le k \le n - m_n$. Modulo $n-1$, we have $a \equiv a+n-1 \equiv k$, $a+1 \equiv a+n \equiv k+1$, ..., $a+m_n-1 \equiv a+n+m_n-2 \equiv k+m_n-1$. Since each of these is an n-riven number and $k+m_n-1 \le n-1$, we can apply Lemma 6 to get that each of k, $k+1, \ldots, k+m_n-1$ divides $n-1$. There are m_n consecutive numbers in this list. Therefore, m_n divides one of them, and thus m_n divides $n-1$. But this contradicts the definition of m_n.

Case 2. $n - m_n < k < n-1$. Since $k+1 \le n-1$, $a+(n-1)-(k+1)$ is in the sequence, and since $2n-k-3 < n+m_n-2$, $a+2(n-1)-(k+1)$ is in the sequence. But each of these in congruent to -1 modulo $n-1$, so we have a contradiction to Corollary 7.

Therefore, $S_n(a) = n-1$.

Now, suppose that $a+n+m_n-1$ is also n-riven. Then $a+m_n$ and $a+m_n+(n-1)$ are both in the sequence. So, $S_n(a+m_n) = m_n$ divides $n-1$, by Lemma 6, contradicting the definition of m_n.

We now construct an infinite family of sequences of n-riven numbers that are of length $n+m_n-1$, thus proving that the bound in Theorem 9 is optimal. One of these sequences, we will prove, is minimal in that there exist no smaller numbers forming an n-riven number sequence of maximal length.

Theorem 10: Fix $\ell = \text{lcm}(1, 2, 3, \ldots, n-1)$ and let a be any integral multiple of ℓ. Then a, $a+1$, $a+2, \ldots, a+n+m_n-2$ is a sequence of consecutive n-riven numbers of maximal length. Further, ℓ is minimal such that ℓ, $\ell+1$, $\ell+2, \ldots,$ $\ell+n+m_n-2$ is a sequence of consecutive n-riven numbers of maximal length.

Proof: We first show that each of these numbers is n-riven. Since $(n-1) \mid a$, it is n-riven, by Corollary 4. For $1 \le t \le n-1$, $S_n(a+t) = t$, which divides a and therefore $a+t$ Thus, $a+t$ is n-riven. Finally, for $1 \le t \le m_n - 1$, $S_n(a+n-1+t) = t$ which, as above, divides $a+t$. Further, by definition of m_n, t divides $n-1$. Hence, $t \mid (a+n-1+t)$ and so $a+n-1+t$ is an n-riven number.

It remains to show that ℓ is the smallest number initiating a maximal sequence of consecutive n-riven numbers. Let $a, a+1, a+2, \ldots, a+n+m_n-2$ be such a sequence. Then, by Theorem 9, $a \equiv 0 \pmod{n-1}$ and so $S_n(a) = n-1$. For all $1 \le t \le n-1$, $a+t$ is an n-riven number, implying that $t \mid (a+t)$ and so $t \mid a$. Thus, $\text{lcm}(1, 2, 3, \ldots, n-1) \mid a$. The result now follows trivially.

REFERENCES

1. C. Cooper & R. E. Kennedy. "On Consecutive Niven Numbers." *The Fibonacci Quarterly* **31.2** (1993):146-51.
2. H. G. Grundman. "Sequences of Consecutive n-Niven Numbers." *The Fibonacci Quarterly* **32.2** (1994):174-75.
3. R. Kennedy, T. Goodman, & C. Best. "Mathematical Discovery and Niven Numbers." *The MATYC Journal* **14** (1980):21-25.
4. B. Wilson. "Construction of Small Consecutive Niven Numbers." *The Fibonacci Quarterly* **34.3** (1996):240-43.
5. B. Wilson. "Construction of $2*n$ Consecutive Niven Numbers." *The Fibonacci Quarterly* **35.2** (1997):122-28.

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