RECIPROCAL SUMS OF SECOND-ORDER RECURRENT SEQUENCES

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1. INTRODUCTION

Let $\mathbb{Z}$ and $\mathbb{R}$ ($\mathbb{C}$) denote the ring of the integers and the field of real (complex) numbers, respectively. For a field F, we put $F^* = F \setminus \{0\}$. Fix $A \in \mathbb{C}$ and $B \in \mathbb{C}^*$, and let $\mathscr{L}(A, B)$ consist of all those second-order recurrent sequences $\{w_n\}_{n\in\mathbb{Z}}$ of complex numbers satisfying the recursion:

$$w_{n+1} = Aw_n - Bw_{n-1} \text{ (i.e., } Bw_{n-1} = Aw_n - w_{n+1}) \text{ for } n = 0, \pm 1, \pm 2, \ldots. \tag{1}$$

For sequences in $\mathscr{L}(A, B)$, the corresponding characteristic equation is $x^2 - Ax + B = 0$, whose roots $(A \pm \sqrt{A^2 - 4B})/2$ are denoted by α and β. If $A \in \mathbb{R}^*$ and $\Delta = A^2 - 4B \geq 0$, then we let

$$\alpha = \frac{A - \text{sg}(A)\sqrt{\Delta}}{2} \quad \text{and} \quad \beta = \frac{A + \text{sg}(A)\sqrt{\Delta}}{2} \tag{2}$$

where $\text{sg}(A) = 1$ if $A > 0$, and $\text{sg}(A) = -1$ if $A < 0$. In the case $w_1 = \alpha w_0$, it is easy to see that $w_n = \alpha^n w_0$ for any integer n. If $A = 0$, then $w_{2n} = (-B)^n w_0$ and $w_{2n+1} = (-B)^n w_1$ for all $n \in \mathbb{Z}$. The Lucas sequences $\{u_n\}_{n\in\mathbb{Z}}$ and $\{v_n\}_{n\in\mathbb{Z}}$ in $\mathscr{L}(A, B)$ take special values at $n = 0, 1$, namely,

$$u_0 = 0,\ u_1 = 1,\ v_0 = 2,\ v_1 = A. \tag{3}$$

It is well known that

$$(\alpha - \beta)u_n = \alpha^n - \beta^n \quad \text{and} \quad v_n = \alpha^n + \beta^n \ \text{ for } n \in \mathbb{Z}. \tag{4}$$

If $A = 1$ and $B = -1$, then those $F_n = u_n$ and $L_n = v_n$ are called Fibonacci numbers and Lucas numbers, respectively.

Let m be a positive integer. In 1974, I. J. Good [2] showed that

$$\sum_{n=0}^{m} \frac{1}{F_{2^n}} = 3 - \frac{F_{2^m-1}}{F_{2^m}}, \quad \text{i.e.,} \quad \sum_{n=0}^{m-1} \frac{(-1)^{2^n}}{F_{2^{n+1}}} = -\frac{F_{2^m-1}}{F_{2^m}};$$

V. E. Hoggatt, Jr., and M. Bicknell [4] extended this by evaluating $\sum_{n=0}^{m} F_{k2^n}^{-1}$, where k is a positive integer. In 1977, W. E. Greig [3] was able to determine the sum $\sum_{n=0}^{m} u_{k2^n}^{-1}$ with $B = -1$; in 1995, R. S. Melham and A. G. Shannon [5] gave analogous results in the case $B = 1$. In 1990, R. André-Jeannin [1] calculated $\sum_{n=1}^{\infty} 1/(u_{kn}u_{k(n+1)})$ and $\sum_{n=1}^{\infty} 1/(v_{kn}v_{k(n+1)})$ in the case $B = -1$ and

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$2\nmid k$, using the Lambert series $L(x)=\sum_{n=1}^{\infty}x^n/(1-x^n)$ $(|x|<1)$; in 1995, Melham and Shannon [5] computed the sums in the case $B=1$, in terms of α and β.

In the present paper we obtain the following theorems that imply all of the above.

Theorem 1: Let m be a positive integer, and f a function such that $f(n)\in\mathbb{Z}$ and $w_{f(n)}\neq 0$ for all $n=0,1,\ldots,m$. Then

$$\sum_{n=0}^{m-1}\frac{B^{f(n)}u_{\Delta f(n)}}{w_{f(n)}w_{f(n+1)}}=\frac{B^{f(0)}u_{f(m)-f(0)}}{w_{f(0)}w_{f(m)}}, \tag{5}$$

where $\Delta f(n)=f(n+1)-f(n)$. If $w_1\neq\alpha w_0$, then

$$\sum_{n=0}^{m-1}\frac{(-1)^n}{w_{f(n)}}\left(\frac{2\alpha^{f(n)}}{w_1-\alpha w_0}-\frac{B^{f(n)}u_{\Delta f(n)}}{w_{f(n+1)}}\right)=\frac{1}{w_1-\alpha w_0}\left(\frac{\alpha^{f(0)}}{w_{f(0)}}-(-1)^m\frac{\alpha^{f(m)}}{w_{f(m)}}\right). \tag{6}$$

Theorem 2: Suppose that $A,B\in\mathbb{R}^*$ and $\Delta=A^2-4B\geq 0$. Let $f:\{0,1,2,\ldots\}\to\{k\in\mathbb{Z}:w_k\neq 0\}$ be a function such that $\lim_{n\to+\infty}f(n)=+\infty$. If $w_1\neq\alpha w_0$, then we have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{B^{f(n)}u_{\Delta f(n)}}{w_{f(n)}w_{f(n+1)}}&=\frac{\alpha^{f(0)}}{(w_1-\alpha w_0)w_{f(0)}}\\&=\sum_{n=0}^{\infty}\frac{(-1)^n}{w_{f(n)}}\left(\frac{2\alpha^{f(n)}}{w_1-\alpha w_0}-\frac{B^{f(n)}u_{\Delta f(n)}}{w_{f(n+1)}}\right).\end{aligned} \tag{7}$$

In the next section we will derive several results from these theorems. Theorems 1 and 2 are proved in Section 3.

2. CONSEQUENCES OF THEOREMS 1 AND 2

Theorem 3: Let k and l be integers such that $w_{kn+l}\neq 0$ for all $n=0,1,2,\ldots$. Then

$$u_k\sum_{n=0}^{m-1}\frac{B^{kn}}{w_{kn+l}w_{k(n+1)+l}}=\frac{u_{km}}{w_l w_{km+l}}\quad\text{for all } m=1,2,3,\ldots. \tag{8}$$

If $A,B\in\mathbb{R}^*$, $A^2\geq 4B$, $k>0$, and $w_1\neq\alpha w_0$, then

$$\sum_{n=0}^{\infty}\frac{u_k B^{kn+l}}{w_{kn+l}w_{k(n+1)+l}}=\frac{\alpha^l}{(w_1-\alpha w_0)w_l} \tag{9}$$

and

$$\sum_{n=0}^{\infty}\left(2\frac{(-\alpha^k)^n}{w_{kn+l}}-(w_1-\alpha w_0)u_k\beta^l\frac{(-B^k)^n}{w_{kn+l}w_{k(n+1)+l}}\right)=\frac{1}{w_l}. \tag{10}$$

Proof: Simply apply Theorems 1 and 2 with $f(n)=kn+l$.

Remark 1: When $B=1$, $l=k$, and $\{w_n\}=\{u_n\}$ or $\{v_n\}$, Melham and Shannon [5] obtained (8) with the right-hand side replaced by a complicated expression in terms of α and β.

Theorem 4: Let $A,B\in\mathbb{R}^*$ and $\Delta=A^2-4B>0$. Then, for any positive integer k, we have

$$\sum_{n=1}^{\infty}\frac{(-B^k)^n}{u_{kn}u_{k(n+1)}}=\frac{\alpha^k}{u_k^2}+\mathrm{sg}(A)\frac{\sqrt{\Delta}}{u_k}\left(4L\left(\frac{\alpha^{4k}}{B^{2k}}\right)-2L\left(\frac{\alpha^{2k}}{B^k}\right)\right) \tag{11}$$

and

$$\sum_{n=1}^{\infty}\frac{(-B^k)^n}{v_{kn}v_{k(n+1)}}=\frac{\mathrm{sg}(A)}{\sqrt{\Delta}}\left(\frac{\alpha^k}{u_{2k}}-\frac{2}{u_k}\left(4L\left(\frac{\alpha^{8k}}{B^{4k}}\right)-4L\left(\frac{\alpha^{4k}}{B^{2k}}\right)+L\left(\frac{\alpha^{2k}}{B^k}\right)\right)\right). \tag{12}$$

Proof: Clearly, $|\alpha|<|\beta|$ and $\beta-\alpha=\mathrm{sg}(A)\sqrt{\Delta}$. Thus, $u_n=(\beta^n-\alpha^n)/(\beta-\alpha)$ and $v_n=\alpha^n+\beta^n$ are nonzero for all $n\in\mathbb{Z}\setminus\{0\}$. Obviously $u_1-\alpha u_0=1$ and $v_1-\alpha v_0=A-2\alpha=\beta-\alpha=\mathrm{sg}(A)\sqrt{\Delta}$. Applying Theorem 3 with $l=k$ and $\{w_n\}_{n\in\mathbb{Z}}=\{u_n\}_{n\in\mathbb{Z}}$ or $\{v_n\}_{n\in\mathbb{Z}}$, we then obtain,

$$\sum_{n=1}^{\infty}\left(u_k\frac{(-B^k)^n}{u_{kn}u_{k(n+1)}}-2\frac{(-\alpha^k)^n}{u_{kn}}\right)=\frac{\alpha^k}{u_k}$$

and

$$\sum_{n=1}^{\infty}\left(u_k\frac{(-B^k)^n}{v_{kn}v_{k(n+1)}}-\frac{2}{\mathrm{sg}(A)\sqrt{\Delta}}\cdot\frac{(-\alpha^k)^n}{v_{kn}}\right)=\frac{\alpha^k/v_k}{\mathrm{sg}(A)\sqrt{\Delta}}.$$

Clearly,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-\alpha^k)^n}{u_{kn}}&=\sum_{n=1}^{\infty}(\beta-\alpha)\frac{(-\alpha^k)^n}{\beta^{kn}-\alpha^{kn}}=(\beta-\alpha)\sum_{n=1}^{\infty}\frac{(-1)^n(\alpha/\beta)^{kn}}{1-(\alpha/\beta)^{kn}}\\&=(\beta-\alpha)\left(2\sum_{\substack{n=1\\2|n}}^{\infty}\frac{(\alpha/\beta)^{kn}}{1-(\alpha/\beta)^{kn}}-\sum_{n=1}^{\infty}\frac{(\alpha/\beta)^{kn}}{1-(\alpha/\beta)^{kn}}\right)\\&=(\beta-\alpha)\left(2L\left(\frac{\alpha^{2k}}{\beta^{2k}}\right)-L\left(\frac{\alpha^k}{\beta^k}\right)\right)=\mathrm{sg}(A)\sqrt{\Delta}\left(2L\left(\frac{\alpha^{4k}}{B^{2k}}\right)-L\left(\frac{\alpha^{2k}}{B^k}\right)\right).\end{aligned}$$

If $|x|<1$, then

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^n\frac{x^n}{1+x^n}&=2\sum_{n=1}^{\infty}\frac{x^{2n}}{1+x^{2n}}-\sum_{n=1}^{\infty}\frac{x^n}{1+x^n}\\&=2\sum_{n=1}^{\infty}\left(\frac{x^{2n}}{1-x^{2n}}-\frac{2x^{4n}}{1-x^{4n}}\right)-\sum_{n=1}^{\infty}\left(\frac{x^n}{1-x^n}-\frac{2x^{2n}}{1-x^{2n}}\right)\\&=2L(x^2)-4L(x^4)-L(x)+2L(x^2)=-4L(x^4)+4L(x^2)-L(x).\end{aligned}$$

Thus,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-\alpha^k)^n}{v_{kn}}&=\sum_{n=1}^{\infty}\frac{(-\alpha^k)^n}{\alpha^{kn}+\beta^{kn}}=\sum_{n=1}^{\infty}(-1)^n\frac{(\alpha/\beta)^{kn}}{1+(\alpha/\beta)^{kn}}\\&=-4L\left(\frac{\alpha^{4k}}{\beta^{4k}}\right)+4L\left(\frac{\alpha^{2k}}{\beta^{2k}}\right)-L\left(\frac{\alpha^k}{\beta^k}\right)\\&=-4L\left(\frac{\alpha^{8k}}{B^{4k}}\right)+4L\left(\frac{\alpha^{4k}}{B^{2k}}\right)-L\left(\frac{\alpha^{2k}}{B^k}\right).\end{aligned}$$

Combining the above and noting that $u_kv_k=u_{2k}$, we then obtain the desired (11) and (12).

Remark 2: If $|x|<1$ then

$$\begin{aligned}L(-x)&=\sum_{n=1}^{\infty}\frac{x^{2n}}{1-x^{2n}}-\sum_{n=1}^{\infty}\frac{x^n}{1+x^n}+\sum_{n=1}^{\infty}\frac{x^{2n}}{1+x^{2n}}\\&=L(x^2)-(L(x)-2L(x^2))+(L(x^2)-2L(x^4))=-2L(x^4)+4L(x^2)-L(x).\end{aligned}$$

Thus, Theorem 2 of André-Jeannin [1] is essentially our (11) and (12) in the special case $B=-1$ and $2\nmid k$.

Theorem 5: Let $k,\ l,\ m\in\mathbb{Z}$ and $l,\ m>0$. If $w_{\binom{k+n}{l}}\neq 0$ for all $n=0,1,\ldots,m$, then

$$\sum_{n=0}^{m-1}\frac{B^{\binom{k+n}{l}}u_{\binom{k+n}{l-1}}}{w_{\binom{k+n}{l}}w_{\binom{k+n+1}{l}}}=\frac{B^{\binom{k}{l}}u_{\binom{k+m}{l}-\binom{k}{l}}}{w_{\binom{k}{l}}w_{\binom{k+m}{l}}}. \tag{13}$$

Proof: Let $f(n)=\binom{k+n}{l}$ for $n\in\mathbb{Z}$. It is well known that $\Delta f(n)=\binom{k+n+1}{l}-\binom{k+n}{l}=\binom{k+n}{l-1}$. So Theorem 5 follows from Theorem 1.

Remark 3: In the case $k=0$ and $l=2$, (13) says that

$$\sum_{n=0}^{m-1}\frac{u_nB^{n(n-1)/2}}{w_{n(n-1)/2}w_{n(n+1)/2}}=\frac{u_{m(m-1)/2}}{w_0w_{m(m-1)/2}}. \tag{14}$$

Theorem 6: Let $a,\ k$ be integers, and m a positive integer. Suppose that $w_{ka^n}\neq 0$ for each $n=0,$ $1,\ldots,m$. Then

$$\sum_{n=0}^{m-1}\frac{B^{ka^n}u_{k(a-1)a^n}}{w_{ka^n}w_{ka^{n+1}}}=\frac{B^ku_{k(a^m-1)}}{w_kw_{ka^m}}. \tag{15}$$

Proof: Just put $f(n)=ka^n$ in Theorem 1.

Remark 4: In the case $a=2$ and $\{w_n\}=\{u_n\}$, (15) becomes

$$\sum_{n=0}^{m-1}\frac{B^{k2^n}}{u_{k2^{n+1}}}=\frac{B^ku_{k(2^m-1)}}{u_ku_{k2^m}}. \tag{16}$$

This was obtained by Melham and Shannon [5] in the case $B=1$ and $k>0$. In the case $a=3$ and $\{w_n\}=\{v_n\}$, (15) turns out to be

$$\sum_{n=0}^{m-1}\frac{B^{k3^n}u_{k3^n}}{v_{k3^{n+1}}}=\frac{B^ku_{k(3^m-1)}}{v_kv_{k3^m}} \tag{17}$$

since $u_{2h}=u_hv_h$ for $h\in\mathbb{Z}$.

Theorem 7: Let k be an integer and m a positive integer. If $w_{k(2^n-1)}\neq 0$ for each $n=0,1,\ldots,$ m, then

$$\sum_{n=0}^{m-1}\frac{B^{k(2^n-1)}u_{k2^n}}{w_{k(2^n-1)}w_{k(2^{n+1}-1)}}=\frac{u_{k(2^m-1)}}{w_0w_{k(2^m-1)}}. \tag{18}$$

Proof: Just apply Theorem 1 with $f(n)=k(2^n-1)$.

3. PROOFS OF THEOREMS 1 AND 2

Lemma 1: For $k, l, m \in \mathbb{Z}$, we have

$$w_k u_{l+m} - w_{k+m} u_l = B^l w_{k-l} u_m \tag{19}$$

and

$$w_k \alpha^l - w_l \alpha^k = (w_1 - \alpha w_0) B^l u_{k-l}. \tag{20}$$

Proof: (i) Fix $k, l \in \mathbb{Z}$. Observe that

$$\begin{pmatrix} w_{k+1} & w_k \\ u_{l+1} & u_l \end{pmatrix} = \begin{pmatrix} w_k & w_{k-1} \\ u_l & u_{l-1} \end{pmatrix} \begin{pmatrix} A & 1 \\ -B & 0 \end{pmatrix}$$
$$= \begin{pmatrix} w_{k-1} & w_{k-2} \\ u_{l-1} & u_{l-2} \end{pmatrix} \begin{pmatrix} A & 1 \\ -B & 0 \end{pmatrix}^2 = \cdots = \begin{pmatrix} w_{k-l+1} & w_{k-l} \\ u_1 & u_0 \end{pmatrix} \begin{pmatrix} A & 1 \\ -B & 0 \end{pmatrix}^l.$$

Taking the determinants, we then get that

$$\begin{vmatrix} w_{k+1} & w_k \\ u_{l+1} & u_l \end{vmatrix} = \begin{vmatrix} w_{k-l+1} & w_{k-l} \\ 1 & 0 \end{vmatrix} \times \begin{vmatrix} A & 1 \\ -B & 0 \end{vmatrix}^l,$$

i.e., $w_k u_{l+1} - w_{k+1} u_l = B^l w_{k-l}$. Thus, (19) holds for $m = 0, 1$.

Each side of (19) can be viewed as a sequence in $\mathcal{L}(A, B)$ with respect to the index m. By induction, (19) is valid for every $m = 0, 1, 2, \ldots$; also (19) holds for each $m = -1, -2, -3, \ldots$. Thus, (19) holds for any $m \in \mathbb{Z}$.

(ii) By induction on l, we find that $w_{l+1} - \alpha w_l = (w_1 - \alpha w_0)\beta^l$. Clearly, both sides of (20) lie in $\mathcal{L}(A, B)$ with respect to the index k. Note that, if $k = l$, then both sides of (20) are zero. As

$$(w_1 - \alpha w_0) B^l = (w_1 - \alpha w_0)\beta^l \alpha^l = (w_{l+1} - \alpha w_l)\alpha^l = \alpha^l w_{l+1} - \alpha^{l+1} w_l,$$

(20) also holds for $k = l + 1$. Therefore, (20) is always valid and we are done.

Proof of Theorem 1: Let $d \in \mathbb{Z}$. In view of Lemma 1, for $n = 0, 1, \ldots, m-1$, we have

$$\frac{u_{d+f(n+1)}}{w_{f(n+1)}} - \frac{u_{d+f(n)}}{w_{f(n)}} = \frac{u_{d+f(n+1)} w_{f(n)} - u_{d+f(n)} w_{f(n+1)}}{w_{f(n)} w_{f(n+1)}}$$
$$= \frac{w_{f(n)} u_{d+f(n)+\Delta f(n)} - w_{f(n)+\Delta f(n)} u_{d+f(n)}}{w_{f(n)} w_{f(n+1)}} = \frac{B^{d+f(n)} w_{-d} u_{\Delta f(n)}}{w_{f(n)} w_{f(n+1)}}.$$

It follows that

$$\sum_{n=0}^{m-1} \frac{B^{d+f(n)} w_{-d} u_{\Delta f(n)}}{w_{f(n)} w_{f(n+1)}} = \sum_{n=0}^{m-1} \left(\frac{u_{d+f(n+1)}}{w_{f(n+1)}} - \frac{u_{d+f(n)}}{w_{f(n)}} \right) = \frac{u_{d+f(m)}}{w_{f(m)}} - \frac{u_{d+f(0)}}{w_{f(0)}}$$

and that

$$\sum_{n=0}^{m-1} (-1)^{n+1} \frac{B^{d+f(n)} w_{-d} u_{\Delta f(n)}}{w_{f(n)} w_{f(n+1)}} = \sum_{n=0}^{m-1} \left((-1)^{n+1} \frac{u_{d+f(n+1)}}{w_{f(n+1)}} + (-1)^n \frac{u_{d+f(n)}}{w_{f(n)}} \right)$$
$$= 2\sum_{n=0}^{m-1} (-1)^n \frac{u_{d+f(n)}}{w_{f(n)}} + (-1)^m \frac{u_{d+f(m)}}{w_{f(m)}} - (-1)^0 \frac{u_{d+f(0)}}{w_{f(0)}}.$$

Putting $d = -f(0)$, we then obtain (5) and

$$\sum_{n=0}^{m-1}(-1)^{n+1}w_{f(0)}\frac{B^{f(n)}u_{\Delta f(n)}}{w_{f(n)}w_{f(n+1)}}=2\sum_{n=0}^{m-1}(-1)^n\frac{B^{f(0)}u_{f(n)-f(0)}}{w_{f(n)}}+(-1)^m\frac{B^{f(0)}u_{f(m)-f(0)}}{w_{f(m)}}.$$

Now suppose that $w_1 \neq \alpha w_0$. By Lemma 1, for each $n = 0, 1, \ldots, m$,

$$\alpha^{f(0)}w_{f(n)}-\alpha^{f(n)}w_{f(0)}=(w_1-\alpha w_0)B^{f(0)}u_{f(n)-f(0)},$$

i.e.,

$$-\frac{B^{f(0)}u_{f(n)-f(0)}}{w_{f(n)}}=\frac{\alpha^{f(n)}w_{f(0)}}{(w_1-\alpha w_0)w_{f(n)}}-\frac{\alpha^{f(0)}}{w_1-\alpha w_0}.$$

Thus,

$$w_{f(0)}\sum_{n=0}^{m-1}(-1)^n\frac{B^{f(n)}u_{\Delta f(n)}}{w_{f(n)}w_{f(n+1)}}$$

$$=2\sum_{n=0}^{m-1}(-1)^n\left(\frac{w_{f(0)}\alpha^{f(n)}}{(w_1-\alpha w_0)w_{f(n)}}-\frac{\alpha^{f(0)}}{w_1-\alpha w_0}\right)+(-1)^m\left(\frac{w_{f(0)}\alpha^{f(m)}}{(w_1-\alpha w_0)w_{f(m)}}-\frac{\alpha^{f(0)}}{w_1-\alpha w_0}\right)$$

and hence

$$\sum_{n=0}^{m-1}\frac{(-1)^n}{w_{f(n)}}\left(\frac{2\alpha^{f(n)}}{w_1-\alpha w_0}-\frac{B^{f(n)}u_{\Delta f(n)}}{w_{f(n+1)}}\right)$$

$$=\frac{2}{w_1-\alpha w_0}\sum_{n=0}^{m-1}(-1)^n\frac{\alpha^{f(0)}}{w_{f(0)}}+\frac{(-1)^m}{w_1-\alpha w_0}\left(\frac{\alpha^{f(0)}}{w_{f(0)}}-\frac{\alpha^{f(m)}}{w_{f(m)}}\right)$$

$$=\frac{1}{w_1-\alpha w_0}\left(\frac{\alpha^{f(0)}}{w_{f(0)}}-(-1)^m\frac{\alpha^{f(m)}}{w_{f(m)}}\right).$$

This proves (6).

Lemma 2: Let $A, B \in \mathbb{R}^*$ and $\Delta = A^2 - 4B \geq 0$. Then

$$\lim_{n\to+\infty}\frac{\alpha^n}{u_n}=0 \tag{21}$$

and

$$\lim_{n\to+\infty}\frac{w_n}{u_{m+n}}=\frac{w_1-\alpha w_0}{\beta^m}\quad\text{for any } m\in\mathbb{Z}. \tag{22}$$

Proof: When $\Delta = 0$ (i.e., $\alpha = \beta$), by induction $u_n = n(A/2)^{n-1}$ for all $n \in \mathbb{Z}$; thus, $u_n \neq 0$ for $n = \pm 1, \pm 2, \pm 3, \ldots$,

$$\lim_{n\to+\infty}\frac{\alpha^n}{u_n}=\lim_{n\to+\infty}\frac{(A/2)^n}{n(A/2)^{n-1}}=0$$

and

$$\lim_{n\to+\infty}\frac{u_{m+n}}{u_n}=\lim_{n\to+\infty}\frac{(m+n)(A/2)^{m+n-1}}{n(A/2)^{n-1}}=\left(\frac{A}{2}\right)^m=\beta^m.$$

In the case $\Delta > 0$, $|\alpha| < |\beta|$; hence, $u_n = (\alpha^n - \beta^n)/(\alpha - \beta)$ is zero if and only if $n = 0$. Thus,

$$\lim_{n\to+\infty}\frac{\alpha^n}{u_n}=(\alpha-\beta)\lim_{n\to+\infty}\frac{1}{1-(\beta/\alpha)^n}=0.$$

Also

$$\lim_{n\to+\infty}\left(\frac{u_{n+1}}{u_n}-\beta\right)=\lim_{n\to+\infty}\frac{\alpha^{n+1}-\beta^{n+1}-\beta(\alpha^n-\beta^n)}{\alpha^n-\beta^n}=\lim_{n\to+\infty}\frac{\alpha-\beta}{1-(\beta/\alpha)^n}=0.$$

If $m\in\{0,1,2,\dots\}$, then

$$\lim_{n\to+\infty}\frac{u_{m+n}}{u_n}=\lim_{n\to+\infty}\prod_{0\le k<m}\frac{u_{k+n+1}}{u_{k+n}}=\beta^m$$

and

$$\lim_{n\to+\infty}\frac{u_{n-m}}{u_n}=\lim_{n\to+\infty}\frac{u_n}{u_{m+n}}=\beta^{-m}.$$

In view of the above, (21) always holds and $\lim_{n\to+\infty}u_{m+n}/u_n=\beta^m$ for all $m\in\mathbb{Z}$. By Lemma 1, $w_1u_n-w_nu_1=Bw_0u_{n-1}$ for $n\in\mathbb{Z}$. Therefore,

$$\lim_{n\to+\infty}\frac{w_n}{u_n}=w_1-\frac{Bw_0}{\lim_{n\to+\infty}u_n/u_{n-1}}=w_1-\frac{Bw_0}{\beta}=w_1-\alpha w_0,$$

and hence (22) is valid.

Proof of Theorem 2: Assume that $w_1\neq\alpha w_0$. In view of Lemma 2,

$$\lim_{m\to+\infty}\frac{B^{f(0)}u_{f(m)-f(0)}}{w_{f(m)}}=B^{f(0)}\frac{\beta^{-f(0)}}{w_1-\alpha w_0}=\frac{\alpha^{f(0)}}{w_1-\alpha w_0}$$

and

$$\lim_{m\to+\infty}\frac{\alpha^m}{w_m}=\lim_{m\to+\infty}\frac{\alpha^m}{u_m}\times\lim_{m\to+\infty}\frac{u_m}{w_m}=0.$$

Applying Theorem 1, we immediately get (7).

Remark 5: On the condition of Theorem 2, if $w_1=\alpha w_0$, then by checking the proof of Theorem 2 we find that

$$\sum_{n=0}^{\infty}\frac{B^{f(n)}u_{\Delta f(n)}}{w_{f(n)}w_{f(n+1)}}=\infty. \tag{23}$$

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