SOME BASIC LINE-SEQUENTIAL PROPERTIES OF POLYNOMIAL LINE-SEQUENCES

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0. INTRODUCTION

There have been many reports on the properties of various polynomial sequences and their generalizations (see, e.g., [1], [3], [4], [5], [6], and [9] and the references therein). In this paper we shall try to treat some polynomial sequences by virtue of the line-sequential formalism developed earlier. To this end, we choose [9] as the guide of our endeavor and obtain some results of a different variety supplementary to those appearing in the literature. In particular, we treat the Morgan-Voyce (MV) polynomial sequences in some detail (for the origination of the MV polynomials, see the references in [1]) and then apply the method to the Jacobsthal (J) and the Vieta (V) polynomial sequences. Finally, we illustrate applications of these results with some examples. The line-sequential treatments of at least some of the other well-known polynomial sequences are somewhat more complicated, so these and other related matters will be discussed in a later report.

1. MV-POLYNOMIAL LINE-SEQUENCES

For convenience of reference, we recap here some of the basic conventions employed in the line-sequential formalism. A homogeneous second-order line-sequence is represented by

$$\bigcup_{u_0, u_1}(c, b): \ldots, u_{-2}, u_{-1}, [u_0, u_1], u_2, u_3, \ldots, u_n, \ldots, \quad n \in z, u_n \in R, \tag{1.1}$$

where c and b, neither zero, are the anharmonic coefficients of the recurrence relation, $cu_{n-2} + bu_{n-1} = u_n$, and the symbol $[u_0, u_1]$ denotes the generating pair of the line-sequence (see §4 in [7]).

The set of line-sequences (1.1) spans a vector space with the pair of basis vectors:

$$U_{1,0}(c, b): \ldots, (c+b^2)/c^2, -b/c, [1, 0], c, cb, c(c+b^2), \ldots; \tag{1.2a}$$

$$U_{0,1}(c, b): \ldots, (c+b^2)/c^3, -b/c^2, 1/c, [0, 1], b, c+b^2, \ldots \tag{1.2b}$$

(see (4.2) and (4.3) in [7]). For convenience, we describe the pair as being "mutually complementary." A general line-sequence (1.1) is then decomposable into its basis components (see (2.9) in [7]) in the following manner:

$$\bigcup_{u_0, u_1}(c, b) = u_0 U_{1,0}(c, b) + u_1 U_{0,1}(c, b). \tag{1.2c}$$

A word on the nomenclature: to comply to the line-sequential format established previously, the symbols and the names adopted here are necessarily somewhat different from some of the corresponding ones of the polynomial sequences as they appear in the literature. However, this will not cause any confusion, as we shall see. For convenience, we adopt the letter M to denote the MV polynomials that are characterized by the values $b = x+2$ and $c = -1$. For the generating pair [1, 0], we then have what we call the "complementary MV-Fibonacci line-sequence," or, for short, the $M_{1,0}$ line-sequence:

$$M_{1,0}(-1, x+2): \ldots, (x^2+4x+3), x+2, [1, 0], -1, -(x+2), -(x^2+4x+3), \ldots. \tag{1.3a}$$

Let $m_n[1, 0]$ denote the n^{th} term (or element) in $M_{1,0}$, counting from the first member of the basis pair as the zeroth term, that is, $m_0[1, 0] = 1$, and increasing toward the right as the positive direction. Then the parity relation will be shown later to be

$$m_{-n}[1, 0] = -m_{n+2}[1, 0]. \tag{1.3b}$$

Let $M_{1,0}(+)$ denote the positive branch, $n \geq 0$, of the $M_{1,0}$ line-sequence, which is denoted by $\{w_n(1, 0; x+2, 1)\}$ in Horadam's notation. Its coefficients table, adapted to the format employed in [9], is given in Table 1 below. The corresponding table for the negative branch can be inferred from Table 1 by means of the parity relation (1.3b).

TABLE 1. The Coefficients Associated with the $M_{1,0}(+)$ Sequence

n	x^0	x^1	x^2	x^3	x^4	x^5
0	1					
1	0					
2	−1					
3	−2	−1				
4	−3	−4	−1			
5	−4	−10	−6	−1		
6	−5	−20	−21	−8	−1	
7	−6	−35	−56	−36	−10	−1

The complementary line-sequence of (1.3a) is given by

$$M_{0,1}(-1, x+2): \ldots, -(x^2+4x+3), -(x+2), -1, [0, 1], x+2, x^2+4x+3, \ldots, \tag{1.4a}$$

which is the MV-Fibonacci line-sequence, or the $M_{0,1}$ line-sequence for short, the positive branch of which is called the MV Even Fibonacci polynomial sequence in [9]. Its parity relation, according to (4.9) in [7], is given by

$$m_{-n}[0, 1] = -m_n[0, 1]. \tag{1.4b}$$

Clearly, $M_{0,1}$ is the negative of one order translation from $M_{1,0}$, that is,

$$M_{0,1} = -TM_{1,0}, \tag{1.4c}$$

where T denotes the translation operator. In terms of the elements,

$$m_n[0, 1] = -m_{n+1}[1, 0]. \tag{1.4d}$$

Definition 1: We say that a line-sequence B is "translationally dependent" on the line-sequence A if and only if B can be obtained from A by means of some (harmonic or anharmonic combinations of) translation operations on A.

Substituting (1.4d) into (1.4b), we obtain the parity relation (1.3b) for $M_{1,0}$.

The line-sequences $M_{1,0}$ and $M_{0,1}$ then form a pair of orthonormal bases spanning the 2D MV line-sequential vector space. Any line-sequence in this space can then be decomposed into its basis components in the manner according to (1.2c).

Combining the parity relations (1.3b) and (1.4b) with the translation relation (1.4d), we obtain the following set of basis relations between the elements of the two basis line-sequences:

$$m_{-n}[1, 0] = m_{n+1}[0, 1], \tag{1.5a}$$

$$m_{-n}[1, 0] = -m_{-(n+1)}[0, 1]; \tag{1.5b}$$

or

$$m_{-n}[0, 1] = m_{n+1}[1, 0], \tag{1.5c}$$

$$m_{-n}[0, 1] = -m_{-(n-1)}[1, 0]. \tag{1.5d}$$

The $M_{1,1}$ line-sequence, the positive branch of which is named the MV Odd Fibonacci polynomial sequence in [9], is given by

$$M_{1,1}(-1, x+2): \ldots, x^2+3x+1, x+1, [1, 1], x+1, x^2+3x+1, \ldots. \tag{1.6a}$$

It decomposes into its basis components according to (1.2c):

$$M_{1,1} = M_{1,0} + M_{0,1}. \tag{1.6b}$$

Or, in terms of the elements,

$$m_n[1, 1] = m_n[1, 0] + m_n[0, 1]. \tag{1.6c}$$

It is seen that the sum of the corresponding coefficients in Table 1 for $M_{1,0}(+)$ above and Table 2(a) for $M_{0,1}(+)$ in [9] equals the corresponding coefficient in Table 2(b) for $M_{1,1}(+)$ in [9], as can be deduced from (1.6c).

Applying relation (1.4c) to the component equation (1.6b), we obtain the following *translational expression* of $M_{1,1}$ in terms of $M_{1,0}$:

$$M_{1,1} = (I - T)M_{1,0}, \tag{1.6d}$$

where I is the identity operator of translation. In terms of the elements, we have

$$m_n[1, 1] = m_n[1, 0] - m_{n+1}[1, 0]. \tag{1.6e}$$

A look at the relevant terms in Table 1 and in Table 2(b) in [9] bears out this relationship.

Since a line-sequence in the MV space can always be decomposed into its basis components, and since the pair of bases are translationally dependent, all MV line-sequences are translationally dependent on either of the basis line-sequences. Since the two bases (4.2) and (4.3) in [7] for the general case are translationally dependent, the above said property must hold in general. We state this in the form of a theorem.

Theorem 1: All line-sequences defined in a line-sequential vector space are translationally dependent on either basis line-sequence.

Applying (1.5a) and (1.5c) to (1.6c), we obtain the parity rule for $M_{1,1}$,

$$m_{-n}[1, 1] = m_{n+1}[1, 1], \tag{1.6f}$$

a property clearly displayed in (1.6a).

The MV-Lucas line-sequence, the positive branch of which is the MV Even Lucas polynomial sequence according to [9], is given by

$$M_{2,x+2}(-1, x+2): \ldots, x^2+4x+2, x+2, [2, x+2], x^2+4x+2, x^3+6x^2+9x+2, \ldots. \tag{1.7a}$$

Applying the geometrical sequences (1.10a) and (1.10b) to the Binet formula (1.12d), see below, and noting that $\alpha\beta = 1$, it is easy to show that the parity relation among the terms in $M_{2,\,x+2}$ is given by

$$m_{-n}[2, x+2] = m_n[2, x+2], \tag{1.7b}$$

which is clearly displayed in the line-sequence (1.7a).

Decomposing (1.7a) into its basis components, we have

$$M_{2,\,x+2} = 2M_{1,\,0} + (x+2)M_{0,\,1} \tag{1.7c}$$

or, in terms of their elements,

$$m_n[2, x+2] = 2m_n[1, 0] + (x+2)m_n[0, 1]. \tag{1.7d}$$

Applying basis relation (1.5a) with parity relations (1.4b) and (1.7b) to (1.7d) above, we get

$$m_n[2, x+2] = 2m_{n+1}[0, 1] - (x+2)m_n[0, 1]. \tag{1.7e}$$

This is the MV-version of the well-known relation $l_n = 2f_{n+1} - f_n$ between the elements of the Lucas and the Fibonacci sequences.

Applying relation (1.4c) to (1.7c), we obtain

$$M_{2,\,x+2} = [2I - (x+2)T]M_{1,\,0}. \tag{1.7f}$$

This is the *translational representation* of the MV-Lucas line-sequence in terms of its first basis. We say the line-sequence $M_{2,\,x+2}$ is "anharmonically" translationally dependent on the basis $M_{1,\,0}$.

The line-sequential form of $M_{-1,\,1}$, the positive branch of which is called the MV Odd Lucas polynomial sequence in [9], is given by

$$M_{-1,\,1}(-1, x+2) \colon \ldots, -(x^2+5x+5), -(x+3), [-1, 1], x+3, x^2+5x+5, \ldots. \tag{1.8a}$$

Its decomposition is given by

$$M_{-1,\,1} = -M_{1,\,0} + M_{0,\,1}. \tag{1.8b}$$

In terms of the elements,

$$m_n[-1, 1] = -m_n[1, 0] + m_n[0, 1]. \tag{1.8c}$$

It is seen that the sum of the negative of a term in Table 1 above and the corresponding term in Table 2(a) in [9] equals the corresponding term in Table 3(b) in [9], as can be deduced from (1.8c).

Applying the relations (1.5a) and (1.5c) to (1.8c), we find the parity relation for the elements of $M_{-1,\,1}$:

$$m_{-n}[-1, 1] = -m_{n+1}[-1, 1], \tag{1.8d}$$

which is clearly displayed in the line-sequence (1.8a).

Applying the relation (1.4c) to (1.8b), we obtain the following translational representation of $M_{-1,\,1}$ in terms of the first basis $M_{1,\,0}$,

$$M_{-1,\,1} = -(I+T)M_{1,\,0}. \tag{1.8e}$$

The following set of interrelationships among the MV polynomials can be shown to hold:

$$M_{1,0}+M_{-1,1}=M_{0,1}; \tag{1.9a}$$
$$M_{1,1}+M_{-1,1}=2M_{0,1}; \tag{1.9b}$$
$$M_{1,1}+M_{1,0}=M_{2,1}, \tag{1.9c}$$
$$(x+2)M_{1,1}-xM_{1,0}=M_{2,x+2}; \tag{1.9d}$$

and so forth.

The pair of geometrical line-sequences relating to $M_{1,0}$ is given by

$$M_{1,\alpha}(-1,x+2):\ldots,\alpha^{-2},\alpha^{-1},[1,\alpha],\alpha^2,\alpha^3,\ldots, \tag{1.10a}$$

and

$$M_{1,\beta}(-1,x+2):\ldots,\beta^{-2},\beta^{-1},[1,\beta],\beta^2,\beta^3,\ldots, \tag{1.10b}$$

where

$$\alpha=[x+2+(x^2+4x)^{1/2}]/2,\quad \beta=[x+2-(x^2+4x)^{1/2}]/2 \tag{1.11a}$$

are the roots of the generating equation

$$q^2-(x+2)q+1=0. \tag{1.11b}$$

Since $M_{1,\alpha}$ and $M_{1,\beta}$ also form a pair of orthogonal (but not normal) bases of the MV vector space (see §3 in [8]), any MV line-sequence can be expressed as a linear combination of its $M_{1,\alpha}$ and $M_{1,\beta}$ components, which, in a manner of speaking, is just its Binet formula.

Generalizing relation (4.9) in [8] and applying basis decompositions in terms of $M_{1,\alpha}$ and $M_{1,\beta}$, we obtain the following set of Binet's formulas for the family of MV line-sequences:

$$M_{1,0}=(-\beta M_{1,\alpha}+\alpha M_{1,\beta})/(\alpha-\beta); \tag{1.12a}$$
$$M_{0,1}=(M_{1,\alpha}-M_{1,\beta})/(\alpha-\beta); \tag{1.12b}$$
$$M_{1,1}=[(1-\beta)M_{1,\alpha}-(1-\alpha)M_{1,\beta}]/(\alpha-\beta); \tag{1.12c}$$
$$M_{2,x+2}=M_{1,\alpha}+M_{1,\beta}; \tag{1.12d}$$
$$M_{-1,1}=[(1+\beta)M_{1,\alpha}-(1+\alpha)M_{1,\beta}]/(\alpha-\beta). \tag{1.12e}$$

Notice that the form of the Binet formulas (1.12b) and (1.12d) justifies our identifying them as the MV-Fibonacci and MV-Lucas line-sequences, respectively, consistent with works in this area; and as a cross check, multiplying (1.12b) and (1.12d), we obtain, in terms of the elements,

$$m_n[0,1]m_n[2,2+x]=m_{2n}[0,1], \tag{1.13}$$

which is the MV version of the well-known relation $f_nl_n=f_{2n}$ between the Fibonacci and Lucas numbers.

Since, by Theorem 1, a line-sequence can always be translationally represented in terms of either of its bases, and since its basis can always be expressed in terms of the geometrical line-sequence, namely Binet's formula, a line-sequence can always be expressed in terms of the geometrical line-sequence which, naturally, is referred to as its Binet formula. Formulas (1.12c) and (1.12e) are such examples. We state this in the form of a theorem.

Theorem 2: All line-sequences defined in a line-sequential vector space are expressible by means of their respective Binet formulas.

2. THE JACOBSTHAL POLYNOMIAL LINE-SEQUENCES

The Jacobsthal (J) polynomial sequence is characterized by the parameters $b = 1$ and $c = x$. (Here, we adopt the convention used in [9]; for another convention used by Horadam, see [4].) The basis pair is given by

$$J_{0,1}(x, 1)\colon \ldots, -(2x^{-2} + x^{-3}), x^{-1} + x^{-2}, -x^{-1}, [0, 1], x, x, x^2 + x, 2x^2 + x, \ldots, \tag{2.1a}$$

$$J_{0,1}(x, 1)\colon \ldots, -(2x^{-3} + x^{-4}), x^{-2} + x^{-3}, -x^{-2}, x^{-1}, [0, 1], 1, x + 1, 2x + 1, \ldots, \tag{2.1b}$$

where the first one will be referred to as the "complementary J-Fibonacci line-sequence" or $J_{1,0}$ line-sequence for short; the second one is the "J-Fibonacci line-sequence," or $J_{0,1}$ line-sequence whose positive branch is called the J-Fibonacci sequence in [9]. The pair then span the 2D J line-sequential vector space. Obviously, the two basis line-sequences are related translationally,

$$TJ_{1,0} = xJ_{0,1}, \tag{2.2a}$$

or, in terms of the elements,

$$j_{n+1}[1, 0] = xj_n[0, 1]. \tag{2.2b}$$

The parity relation of the terms in $J_{1,0}$ can be shown to be

$$j_{-n}[1, 0] = (-1)^{n+2} x^{-(n+1)} j_{n+2}[1, 0]. \tag{2.3a}$$

According to (4.9) in [7], the parity relation for terms in $J_{0,1}$ is given by

$$j_{-n}[0, 1] = (-1)^{n+1} x^{-n} j_n[0, 1]. \tag{2.3b}$$

Substituting the translation relation (2.2b) into (2.3b), we get (2.3a).

Using these parity relations with the translation relation, we obtain the following set of relations between the elements of the two basis line-sequences:

$$j_{-n}[1, 0] = (-x)^{-n} j_{n+1}[0, 1], \tag{2.4a}$$

$$j_{-n}[1, 0] = xj_{-(n+1)}[0, 1]; \tag{2.4b}$$

or

$$j_{-n}[0, 1] = (-x)^{-(n+1)} j_{n+1}[1, 0], \tag{2.4c}$$

$$j_{-n}[0, 1] = x^{-1} j_{-(n-1)}[1, 0]. \tag{2.4d}$$

The coefficient table of $J_{1,0}(+)$ is given in Table 2 below.

TABLE 2. The Coefficients Associated with the $J_{1,0}(+)$ Sequence

n	x^0	x^1	x^2	x^3
0	1			
1	0			
2	0	1		
3	0	1		
4	0	1	1	
5	0	1	2	
6	0	1	3	1
7	0	1	4	3

The J-Lucas line-sequence is given by

$$J_{2,1}(x,1):\ldots,-3x^{-2}-x^{-3},2x^{-1}+x^{-2},-x^{-1},[2,1],2x+1,3x+1,2x^2+4x+1,\ldots,\qquad(2.5a)$$

which is a linear combination of the basis line-sequences (2.1a) and (2.1b),

$$J_{2,1}=2J_{1,0}+J_{0,1}\qquad(2.5b)$$

or, in terms of the corresponding members in these line-sequences,

$$j_n[2,1]=2j_n[1,0]+j_n[0,1].\qquad(2.5c)$$

It is seen that the sum of twice a term in Table 2 for $J_{1,0}(+)$ above and a term in Table 4(a) for $J_{0,1}(+)$ in [9] equals the corresponding term in Table 4(b) for $J_{2,1}(+)$ in [9], as can be deduced from relation (2.5c) above.

From the Binet formula (2.8c) below, using (2.6a) and (2.6b), noting that $\alpha\beta=-x$, we obtain the following parity relation for the J-Lucas line-sequence:

$$j_{-n}[2,1]=(-1)^n x^{-n} j_n[2,1].\qquad(2.5d)$$

Applying parity relation (2.5d) and relations (2.4a) and (2.4c) to the component equation (2.5c), using the translation equation (2.2b), we obtain

$$j_n[2,1]=2x^{-1}j_{n+2}[1,0]-j_n[0,1],\qquad(2.5e)$$

which is the J-version of the relation $l_n=2f_{n+1}-f_n$.

Applying the translation relation (2.2a) to the basis component equation (2.5b), we obtain the translational representation of the J-Lucas line-sequence in terms of the $J_{1,0}$ basis,

$$J_{2,1}=(2I+x^{-1}T)J_{1,0},\qquad(2.5f)$$

a result consistent with the statement of Theorem 1 above.

The pair of geometrical line-sequences relating to $J_{1,0}$ is given by

$$J_{1,\alpha}(x,1):\ldots,\alpha^{-2},\alpha^{-1},[1,\alpha],\alpha^2,\alpha^3,\ldots,\qquad(2.6a)$$

$$J_{1,\beta}(x,1):\ldots,\beta^{-2},\beta^{-1},[1,\beta],\beta^2,\beta^3,\ldots,\qquad(2.6b)$$

where

$$\alpha=[1+(1+4x)^{1/2}]/2,\quad \beta=[1-(1+4x)^{1/2}]/2\qquad(2.7a)$$

are the roots of the generating equation

$$q^2-q-x=0.\qquad(2.7b)$$

Here, considering the multitude of recurring polynomial sequences which may be treated in this manner, we retain the use of the same pair of letters α and β to represent the roots of the *respective* generating equation of each case, rather than adopt a new pair of letters each time for each case; while the pair of letters A and B remains reserved for representing the large and the (negative) small golden ratios.

Then an arbitrary J line-sequence can be expressed in terms of $J_{1,\alpha}$ and $J_{1,\beta}$. In particular,

$$J_{1,0}=(-\beta J_{1,\alpha}+\alpha J_{1,\beta})/(\alpha-\beta),\qquad(2.8a)$$

which is Binet's formula for the complementary J-Fibonacci line-sequence, and

$$J_{0,1} = (J_{1,\alpha} - J_{1,\beta}) / (\alpha - \beta), \tag{2.8b}$$

which is Binet's formula for the J-Fibonacci line-sequence, and

$$J_{2,1} = J_{1,\alpha} + J_{1,\beta}, \tag{2.8c}$$

which is Binet's formula for the J-Lucas line-sequence.

It is easy to see that

$$j_n[0, 1]\, j_n[2, 1] = j_{2n}[0, 1], \tag{2.9}$$

which is the J-version of the basic relation $f_n l_n = f_{2n}$.

3. THE VIETA POLYNOMIAL LINE-SEQUENCES

The V-polynomial sequence is characterized by the parameters $b = x$ and $c = -1$. Observing that, if we put $x + 2 = x'$ for the MV-polynomials, then the latter will be line-sequentially equivalent to the V-polynomials. Therefore, the line-sequential relations of the V-polynomials can be obtained directly from the corresponding ones of the MV-polynomials. For convenience of reference, however, we compile the following essential relations for the V-polynomials.

The basis pair of the V line-sequences is given by

$$V_{1,0}(-1, x): \ldots, -(2x - x^3), -(1 - x^2), x, [1, 0], -1, -x, 1 - x^2, 2x - x^3, \ldots, \tag{3.1a}$$

$$V_{0,1}(-1, x): \ldots, 2x - x^3, 1 - x^2, -x, -1, [0, 1], x, -(1 - x^2), -(2x - x^3) \ldots, \tag{3.1b}$$

where the first one is the complementary V-Fibonacci line-sequence, or $V_{1,0}$ line-sequence for short; the second is the V-Fibonacci line-sequence, or $V_{0,1}$ line-sequence for short. This pair spans the 2D V line-sequential vector space.

Obviously, we have the following translational relation between the two basis line sequences:

$$V_{0,1} = -TV_{1,0} \tag{3.2a}$$

or, in terms of the elements,

$$v_n[0, 1] = -v_{n+1}[1, 0]. \tag{3.2b}$$

The parity relation of the elements in $V_{1,0}$ is found to be

$$v_{-n}[1, 0] = -v_{n+2}[1, 0]. \tag{3.3a}$$

From (4.9) in [7], the parity relation for the elements in $V_{0,1}$ is found to be

$$v_{-n}[0, 1] = -v_n[0, 1], \tag{3.3b}$$

which is clearly borne out in (3.1b). Applying (3.2b) to (3.3b), we obtain (3.3a).

Using these parity relations together with the translation relation (3.2b), we obtain the following set of relations between the elements of the two basis line-sequences:

$$v_{-n}[1, 0] = v_{n+1}[0, 1], \tag{3.4a}$$

$$v_{-n}[1, 0] = -v_{-(n+1)}[0, 1]; \tag{3.4b}$$

or

$$v_{-n}[0, 1] = v_{n+1}[1, 0], \tag{3.4c}$$

$$v_{-n}[0, 1] = -v_{-(n-1)}[1, 0]. \tag{3.4d}$$

This set of relations parallels exactly the set for the MV line-sequences, namely, from (1.5a) to (1.5d), as it should be.

The coefficient table of $V_{1,0}(+)$ is given in Table 3.

TABLE 3. The Coefficients Associated with the $V_{1,0}(+)$ Sequence

n	x^0	x^1	x^2	x^3	x^4	x^5
0	1					
1	0					
2	−1					
3	0	−1				
4	1	0	−1			
5	0	2	0	−1		
6	−1	0	3	0	−1	
7	0	−3	0	4	0	−1

The coefficient table of $xV_{0,1}(+) = V_{0,x}(+)$ is given in Table 4.

TABLE 4. The Coefficients Associated with the $V_{0,x}(+)$ Sequence

n	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7
0	0							
1	0	1						
2	0	0	1					
3	0	−1	0	1				
4	0	0	−2	0	1			
5	0	1	0	−3	0	1		
6	0	0	3	0	−4	0	1	
7	0	−1	0	6	0	−5	0	1

The V-Lucas line-sequence is given by

$$V_{2,x}(-1, x): \ldots, -x(3-x^2), -(2-x^2), x, [2, x], -(2-x^2), -x(3-x^2), \ldots. \tag{3.5a}$$

The coefficient table of $V_{2,x}(+)$ is given in Table 5.

TABLE 5. The Coefficients Associated with the $V_{2,x}(+)$ Sequence

n	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7
0	2							
1	0	1						
2	−2	0	1					
3	0	−3	0	1				
4	2	0	−4	0	1			
5	0	5	0	−5	0	1		
6	−2	0	9	0	−6	0	1	
7	0	−7	0	14	0	−7	0	1

The decomposition of $V_{2,x}$ into its basis components is given by

$$V_{2,x} = 2V_{1,0} + xV_{0,1} \tag{3.5b}$$

or, in terms of the elements,

$$v_n[2, x] = 2v_n[1, 0] + xv_n[0, 1]. \tag{3.5c}$$

It can be seen that the sum of twice a coefficient in Table 3 and the corresponding coefficient in Table 4 equals the corresponding coefficient in Table 5, as can be deduced from (3.5c).

The parity relation among the terms of $V_{2,x}$ is obtained from (1.7b):

$$v_{-n}[2, x] = v_n[2, x], \tag{3.6a}$$

which is apparent in (3.5a).

The V-version of the relation $l_n = 2f_{n+1} - f_n$ is obtained from (1.7e):

$$v_n[2, x] = 2v_{n+1}[0, 1] - xv_n[0, 1]. \tag{3.6b}$$

The translational expression of $V_{2,x}$ in terms of $V_{1,0}$ is obtained from (1.7f):

$$V_{2,x} = (2I - xT)V_{1,0}. \tag{3.6c}$$

The pair of geometrical line-sequences relating to $V_{1,0}$ is given by

$$V_{1,\alpha}(-1, x): \ldots, \alpha^{-2}, \alpha^{-1}, [1, \alpha], \alpha^2, \alpha^3, \ldots, \tag{3.7a}$$

$$V_{1,\beta}(-1, x): \ldots, \beta^{-2}, \beta^{-1}, [1, \beta], \beta^2, \beta^3, \ldots, \tag{3.7b}$$

where

$$\alpha = [x + (x^2 - 4)^{1/2}]/2, \quad \beta = [x - (x^2 - 4)^{1/2}]/2 \tag{3.8a}$$

are the roots of the generating equation

$$q^2 - xq + 1 = 0. \tag{3.8b}$$

Hence, the Binet formula for the $V_{1,0}$ line-sequence is given by

$$V_{1,0} = (-\beta V_{1,\alpha} + \alpha V_{1,\beta}) / (\alpha - \beta), \tag{3.9a}$$

the Binet formula for the $V_{0,1}$ line-sequence is given by

$$V_{0,1} = (V_{1,\alpha} - V_{1,\beta}) / (\alpha - \beta), \tag{3.9b}$$

and the Binet formula for the V-Lucas line-sequence is given by

$$V_{2,x} = V_{1,\alpha} + V_{1,\beta}. \tag{3.9c}$$

Obviously,

$$v_n[0, 1] v_n[2, x] = v_{2n}[0, 1], \tag{3.9d}$$

which is the V-version of the relation $f_n l_n = f_{2n}$.

4. SOME APPLICATIONS

We illustrate the application of the foregoing results with a few examples.

Example 1: For the MV-Lucas line-sequence, by the rule of line-sequential addition, we have

$$M_{1,x+2} + M_{1,0} = M_{2,x+2}.$$

Using translation relation (1.4c), we obtain $(T - T^{-1})M_{0,1} = M_{2,x+2}$. So, in general, we have

$$(T^{n+1}-T^{n-1})M_{0,1}=T^nM_{2,x+2}. \tag{4.1a}$$

This is the translational representation of the MV-Lucas line-sequence in terms of its second basis. In elements form, this becomes

$$m_{n+1}[0,1]-m_{n-1}[0,1]=m_n[2,x+2], \tag{4.1b}$$

which is the MV-version of the well-known relation between the Fibonacci and the Lucas numbers $f_{n+1}+f_{n-1}=l_n$.

Applying parity relation (1.4b) to (4.1b), we obtain

$$m_{-(n-1)}[0,1]-m_{-(n+1)}[0,1]=m_n[2,x+2] \tag{4.1c}$$

or

$$(T^{-(n-1)}-T^{-(n+1)})M_{0,1}=T^nM_{2,x+2}, \tag{4.1d}$$

which is the *negative* translational representation of the MV-Lucas line-sequence. From (4.1d), it can easily be inferred that

$$(T^{-(n-1)}+T^{-(n+1)})F_{0,1}=(-1)^nT^nF_{2,1}, \tag{4.1e}$$

which is the negative translational representation of the Lucas line-sequence. Therefore, in terms of the elements, we obtain the expression of the Lucas numbers in terms of the Fibonacci numbers with *negative* indices, i.e.,

$$f_{-(n-1)}+f_{-(n+1)}=(-1)^nl_n, \tag{4.1f}$$

which is a particular case of equation (2.16) of Horadam [2].

Example 2: For the J-Lucas line-sequence, we have $J_{1,1}+J_{1,0}=J_{2,1}$. Using translation relation (2.2a), we have $[T+xT^{-1}]J_{0,1}=J_{2,1}$. Hence, we obtain

$$[T^{n+1}+xT^{n-1}]J_{0,1}=T^nJ_{2,1}. \tag{4.2a}$$

This is the translational expression of the J-Lucas line-sequence in terms of its second basis. In the elements form, we have

$$j_{n+1}[0,1]+xj_{n-1}[0,1]=j_n[2,1], \tag{4.2b}$$

which is the J-version of the relation $f_{n+1}+f_{n-1}=l_n$.

Applying parity relation (2.3b) to (4.2b) and using the translation operation, we obtain

$$(-1)^nx^n(xT^{-(n+1)}+T^{-(n-1)})J_{0,1}=T^nJ_{2,1}, \tag{4.2c}$$

which is the negative translational expression of the J-Lucas line-sequence in terms of its second basis.

Example 3: For the V-Lucas line-sequence, we start with $V_{1,x}+V_{1,0}=V_{2,x}$. Using translation relation (3.2a), this becomes $(T-T^{-1})V_{0,1}=V_{2,x}$. Hence, we have

$$(T^{n+1}-T^{n-1})V_{0,1}=T^nV_{2,x}. \tag{4.3a}$$

This is the translational representation of the V-Lucas line-sequence in terms of its second basis. In the elements form, we find that

$$v_{n+1}[0,1]-v_{n-1}[0,1]=v_n[2,x], \tag{4.3b}$$

which is the V-version of the relation $f_{n+1}+f_{n-1}=l_n$.

Applying parity relation (3.3b) to (4.3b) and using the translation operation, we have

$$(-T^{-(n+1)}+T^{-(n-1)})V_{0,1}=T^nV_{2,x}, \tag{4.3c}$$

which is the negative translational expression of the V-Lucas line-sequence in terms of its second basis.

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