THE FILBERT MATRIX

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1. INTRODUCTION

The $n \times n$ Hilbert matrix is the $n \times n$ matrix whose (i, j)-entry is $\frac{1}{i+j-1}$. In [1], Man-Duen Choi explores many fascinating properties of the Hilbert matrix, including the fact that the (i, j)-entry of its inverse is

$$\alpha_{ij} = (-1)^{i+j}(i+j-1)\binom{n+i-1}{n-j}\binom{n+j-1}{n-i}\binom{i+j-2}{i-1}^2. \tag{1}$$

Choi asks what sort of coincidence it is if the inverse of a matrix of reciprocals of integers has integer entries. In this paper we show that the inverses of the Hankel matrices based on the reciprocals of the Fibonacci numbers, the reciprocals of the binomial coefficients $\binom{i+j}{2}$, and the reciprocals of the binomial coefficients $\binom{i+j+2}{3}$ all have integer entries. We also find formulas for the entries of the inverses of these matrices and related matrices.

Definition 1.1: Let $\{a_k\}$ be an integer sequence with $a_k \neq 0$ for $k \geq 1$. A *reciprocal Hankel matrix* based on $\{a_k\}$ is a matrix whose (i, j)-entry is $1/a_{i+j-1}$. We denote the $n \times n$ reciprocal Hankel matrix based on $\{a_k\}$ by $R_n(a_k)$.

The formula for the entries of the inverse of $R_n(F_k)$ bears a striking resemblance to the formula for the entries of the inverse of the Hilbert matrix. Therefore, we call a reciprocal Hankel matrix based on the Fibonacci numbers a *Filbert matrix*.

2. FILBERT MATRICES

We need the Fibonomial coefficients to describe the inverse of the Filbert matrix. See [2] for more information on the Fibonomial coefficients.

Definition 2.1: The *Fibonomial coefficients* are

$$\left(\!\binom{n}{k}\!\right) = \prod_{i=1}^{k} \frac{F_{n-i+1}}{F_i},$$

where n and k are nonnegative integers.

Theorem 2.1: Let $e(n, i, j) = n(i+j+1) + \binom{i}{2} + \binom{j}{2} + 1$, and let $W(n)$ be the $n \times n$ matrix whose (i, j)-entry is

$$W_{ij}(n) = (-1)^{e(n,i,j)} F_{i+j-1} \left(\!\binom{n+i-1}{n-j}\!\right)\left(\!\binom{n+j-1}{n-i}\!\right)\left(\!\binom{i+j-2}{i-1}\!\right)^2.$$

Then the $n \times n$ matrix $W(n)$ is the inverse of the Filbert matrix $R_n(F_k)$, and $W(n)$ is an integer matrix.

This theorem is a special case of Theorem 2.2, which we prove below. The formula for the entries of the inverse closely corresponds to the formula for the entries of the inverse of the $n \times n$ Hilbert matrix. It results from (1) by changing all binomial coefficients to Fibonomial coefficients and changing the exponent of -1. The pattern of the signs of entries the inverse of $R_n(F_k)$ is that they are constant on 2×2 blocks, and alternate between blocks.

The Fibonacci polynomials $f_n(x)$ are defined by $f_0(x) = 0$, $f_1(x) = 1$, $f_n(x) = xf_{n-1}(x) + f_{n-2}(x)$ for $n \geq 2$. We also use f_n to denote the Fibonacci polynomial $f_n(x)$, especially when we want to reduce the clutter in some equations. The x-Fibonomial coefficients are the obvious generalization of the Fibonomial coefficients.

Definition 2.2: The *x-Fibonomial coefficients* are

$$\left(\!\!\binom{n}{k}\!\!\right)_x = \prod_{i=1}^{k} \frac{f_{n-i+1}(x)}{f_i(x)},$$

where n and k are nonnegative integers.

To form the (i, j)-entry of the inverse of $R_n(f_k(x))$, replace each Fibonacci number and Fibonomial coefficient in $W_{ij}(n)$ with the corresponding Fibonacci polynomial and x-Fibonomial coefficient.

Theorem 2.2: Let $V(n)$ be the $n \times n$ matrix whose (i, j)-entry is

$$V_{ij}(n) = (-1)^{e(n,i,j)} f_{i+j-1} \left(\!\!\binom{n+i-1}{n-j}\!\!\right)_x \left(\!\!\binom{n+j-1}{n-i}\!\!\right)_x \left(\!\!\binom{i+j-2}{i-1}\!\!\right)_x^2.$$

Then the $n \times n$ matrix $V(n)$ is the inverse of the Filbert matrix $R_n(f_k(x))$, and the entries of $V(n)$ are integer polynomials.

The recurrence

$$\left(\!\!\binom{n}{k}\!\!\right)_x = f_{k-1}(x)\left(\!\!\binom{n-1}{k}\!\!\right)_x + f_{n-k+1}(x)\left(\!\!\binom{n-1}{k-1}\!\!\right)_x$$

shows that the Fibonomial coefficients are integer polynomials, which implies that the entries of $V(n)$ are integer polynomials.

3. TECHNOLOGY

The proof of Theorem 2.2 and proofs of succeeding theorems amount to proving various identities involving sums of products of Fibonomial coefficients and binomial coefficients. We supply computer proofs of these identities. In some cases, the computer cannot do the entire proof directly, and human intervention is required to separate the proof into smaller pieces that can be done by computer.

The program and packages used to produce the proofs for this paper include Maple V Release 5, the Maple package EKHAD written by Doron Zeilberger, and the Mathematica package `MultiSum` written by Kurt Wegschaider. EKHAD is described in [3] and is available through the web site www.math.temple.edu/~zeilberg. `MultiSum` is described in [4] and is available through the web site www.risc.uni-linz.ac.at/software/. The particular functions that we use from these packages are `zeil` from `EKHAD` and `FindRecurrence` from `MultiSum`.

Both of these functions find a telescoped recurrence for a summand $F(n,k)$, where k is the summation variable. The function `zeil` uses Zeilberger's algorithm to find a rational function $R(n,k)$ and a recurrence operator $P(n, N)$, where N is the shift operator in n such that

$$P(n, N)(F(n,k)) = R(n, k+1)F(n, k+1) - R(n,k)F(n,k). \tag{2}$$

Let $f(n)$ be the unrestricted sum $\sum_k F(n,k)$. In many situations, equation (2) implies that $P(n, N)f(n) = 0$, making it easy to verify that $f(n)$ is constant.

The function `FindRecurrence` gives similar results with summands of the form $F(\mathbf{n}, \mathbf{k})$, where $\mathbf{n}$ and $\mathbf{k}$ are vectors.

Maple V Release 5 also includes an implementation of Zeilberger's algorithm as the function `sumrecursion` of the package `sumtools`. However, `sumrecursion` only gives the recurrence operator $P(n, N)$, and not the rational function $R(n,k)$, which will be essential when we prove identities involving a restricted sum.

The sums involved in the proof of Theorem 2.2 are of products of Fibonomials not binomials, so these procedures do not apply. However, we obtained recurrences for sums of products of Fibonomials by modifying recurrences found by these procedures for the corresponding sums of products of binomials.

4. PROOF OF THEOREM 2.2

The (i,m)-entry of the product $V(n)R_n(f_k(x))$ is $p(n,i,m) = \sum_{j=1}^{n} P(n,i,m,j)$, where

$$P(n,i,m,j) = (-1)^{e(n,i,j)} \frac{f_{i+j-1}}{f_{j+m-1}} \left(\!\!\binom{n+i-1}{n-j}\!\!\right)_x \left(\!\!\binom{n+j-1}{n-i}\!\!\right)_x \left(\!\!\binom{i+j-2}{i-1}\!\!\right)_x^2.$$

The summand satisfies the following recurrence relation that is related to a recurrence produced by `FindRecurrence` for an entry of the product of the Hilbert matrix and its inverse.

Lemma 4.1: The summand $P(n,i,m,j)$ satisfies the recurrence relation

$$\begin{aligned} &-f_{n-i+1}f_{n+i-2}(P(n,i-1,m,j) - P(n-1,i-1,m,j)) \\ &\quad + (-1)^{n+i} f_{i-1}^2 (P(n,i,m,j) - P(n-1,i,m,j)) = 0, \end{aligned} \tag{3}$$

and the sum $p(n,i,m)$ satisfies the recurrence relation

$$\begin{aligned} &-f_{n-i+1}f_{n+i-2}(p(n,i-1,m) - p(n-1,i-1,m)) \\ &\quad + (-1)^{n+i} f_{i-1}^2 (p(n,i,m) - p(n-1,i,m)) = 0. \end{aligned} \tag{4}$$

Proof: Write each of the terms in (3) as a multiple of $P(n-1,i-1,m,j)$ to get the equation

$$\begin{aligned} &-f_{n-i+1}f_{n+i-2}(P(n,i-1,m,j) - P(n-1,i-1,m,j)) \\ &\quad + (-1)^{n+i} f_{i-1}^2 (P(n,i,m,j) - P(n-1,i,m,j)) \\ &= \frac{f_{n+i-2}}{f_{n-i+1}f_{n-j}f_{i+j-1}} M(n,i,j)P(n-1,i-1,m,j), \end{aligned} \tag{5}$$

where

$$\begin{aligned} M(n,i,j) = {} & (-1)^{i+j} f_{n+i-1}f_{n+j-1}f_{i+j-2} + f_{n-i}f_{n-j}f_{i+j-2} \\ & + (-1)^{i+j-1} f_{n+i-2}f_{n+j-1}f_{i+j-1} + f_{n-i+1}f_{n-j}f_{i+j-1}. \end{aligned} \tag{6}$$

It suffices to show that $M(n,i,j)=0$. But this follows from the standard Fibonacci identities $f_{n-i}f_{i+j-2}+f_{n-i+1}f_{i+j-1}=f_{n+j-1}$ and $f_{n+i-2}f_{i+j-1}-f_{n+i-1}f_{i+j-2}=(-1)^{i+j-2}f_{n-j}$. □

If we can establish $p(n,1,1)=1$, $p(n,1,m)=0$ if $m\neq 1$, and $p(n,n,n)=1$, then (4) shows that $p(n,i,m)=1$ if $i=m$ and $p(n,i,m)=0$ if $i\neq m$, for $1\le i,m\le n$.

Case $p(n,1,m)$. The summand $P(n,1,m,j)$ satisfies the recurrence

$$\begin{aligned}(-1)^{m+1}f_{n-1}f_{n+m-2}P(n,1,m-1,j)-f_nf_{n-m+1}P(n-1,1,m-1,j)\\+(-1)^m f_{n-1}f_{n+m-1}P(n,1,m,j)+f_nf_{n-m}P(n-1,i,m,j)=0,\end{aligned}\tag{7}$$

and this implies a similar recurrence for $p(n,1,m)$. The proof of (7) is similar to the proof of Lemma 4.1. The initial values of this recurrence are $p(m,1,m)$ and $p(n,1,1)$. The summand $P(m,1,m,j)$ satisfies the recurrence

$$(-1)^m f_mf_{m-1}P(m,1,m,j)=G_1(m,j+1)-G_1(m,j)$$

where $G_1(m,j)=(-1)^{j-1}f_jf_{j-1}P(m,1,m,j)$. Since the support of G_1 is $2\le j\le m$, this equation implies that $(-1)^m f_mf_{m-1}p(m,1,m)=0$. Therefore, when $m>1$ we get $p(m,1,m)=0$. Finally, the summand $P(n,1,1,j)$ satisfies

$$(-1)^n f_n^2P(n,1,1,j)=G_2(n,j+1)-G_2(n,j),$$

where $G_2(n,j)=(-1)^{j-1}f_j^2P(n,1,1,j)$. In this case, the support of G_2 is $1\le j\le n$, so summing over j from 1 to n gives $(-1)^n f_n^2 p(n,1,1)=-G_2(n,1)=(-1)^n f_n^2$, implying $p(n,1,1)=1$.

Case $p(n,n,n)$. The summand $P(n,n,n,j)$ satisfies the recurrence

$$P(n+1,n+1,n+1,j)-P(n,n,n,j)=G_3(n,j+1)-G_3(n,j),$$

where

$$G_3(n,j)=(-1)^{e(n,n,j)}\left(\frac{f_{3n+j-1}}{f_{n+j-1}}+2(-1)^n\right)\left(\binom{2n-1}{n-j+1}\right)_x\left(\binom{n+j-2}{j-2}\right)_x^2.$$

When we sum over j, the right-hand side telescopes to 0 and the left-hand side is $p(n+1,n+1,n+1)-p(n,n,n)$. This completes the proof of Theorem 2.2.

5. RECIPROCAL HANKEL MATRICES BASED ON BINOMIAL COEFFICIENTS

In this section we will prove that certain reciprocal matrices based on binomial coefficients have integer entries. We will give formulas for the entries of the inverses of these matrices.

Let $a_k=\binom{k+1}{2}$.

Theorem 5.1: Let $A(n)$ be the $n\times n$ matrix whose (i,j)-entry is

$$A_{ij}(n)=\sum_{k=0}^{j-1}(-1)^{i+k+1}\binom{n+i}{n-k}\binom{n+k}{n-i}\binom{i+k-1}{k}\binom{i+k}{k}\frac{i}{2}.$$

Then $A_{ij}(n)$ is an integer, and $A(n)$ is the inverse of the matrix $R_n(a_k)$.

Proof: First, we show that $A_{ij}(n)$ is an integer. We use the well-known fact that, if a is even and b is odd, then $\binom{a}{b}$ is even. If i is even, then obviously $A_{ij}(n)$ is an integer, so assume that i is odd. Now, if k is also odd, then $\binom{i+k}{k}$ is even, so we may assume that k is even. Now, one of $\binom{n+i}{i+k}$ and $\binom{n+k}{i+k}$ is even.

Theorem 5.2 below shows that $A(n)$ is the inverse of the matrix $R_n(a_k)$. □

Let $b_k = b_k(r)$ be the binomial coefficient $\binom{k+r-1}{r}$. Suppose that r is a positive integer and $r \geq 3$. Then the inverse of $R_n(b_k(r))$ does not always have integer entries, but the values of n for which the inverse does have integer entries seem to occur periodically. Further, when the entries are not integers, the denominators are divisors of r. The following conjecture is true for $n \leq 20$, $r \leq 10$, and r an integer.

Conjecture 5.1: Suppose that r is a positive integer. The inverse of the matrix $R_n(b_k(r))$ has integer entries if and only if $n \equiv 0 \pmod{q}$ or $n \equiv 1 \pmod{q}$ for all prime powers q that divide r.

We do have an explicit formula for the entries of the inverse.

Theorem 5.2: Let $B(n,r)$ be the $n \times n$ matrix whose (i, j)-entry is

$$B_{ij}(n,r) = \sum_{k=0}^{j-1}(-1)^{i+k+1}\binom{n+i+r-2}{i}\binom{n}{i}\binom{n+k+r-2}{k}\binom{n}{k}\frac{i^2\prod_{l=0}^{r-3} i+j+l}{r\prod_{l=0}^{r-2} i+k+l}.$$

Then $B(n,r)$ is the inverse of the matrix $R_n(b_k)$.

The theorem is valid if r is an indeterminate, not just if it is a positive integer. Also note that $B_{ij}(n,1)$ simplifies to α_{ij}, the (i, j)-entry of the inverse of the Hilbert matrix, and $B_{ij}(n,2)$ is equal to $A_{ij}(n)$.

Proof: Let

$$H(n,i,m,j,k) = (-1)^{i+k+1}\binom{n+i+r-2}{i}\binom{n}{i} \times \binom{n+k+r-2}{k}\binom{n}{k}\frac{i^2\prod_{l=0}^{r-3} i+j+l}{r\prod_{l=0}^{r-2} i+k+l}\frac{1}{\binom{j+m+r-2}{r}},$$

so that $h(n,i,m) = \sum_{j=1}^{n}\sum_{k=0}^{j-1} H(n,i,m,j,k)$ is the (i,m)-entry of $B(n,r)R_n(b_k)$. Then H satisfies the recurrence

$$\begin{aligned}
&n^2(i-m+r-1)(n-i+r-1)(n+i+r-3)H(n-1,i-1,m-1,j,k)\\
&-n^2(i-m-1)(n-i+r-1)(n+i+r-3)H(n-1,i-1,m,j,k)\\
&+n^2(i-1)^2(i-m+1)H(n-1,i,m-1,j,k)\\
&-n^2(i-1)^2(i-m-r+1)H(n-1,i,m,j,k)\\
&-(n+r-2)^2(i-m+r-1)(n-i+1)(n+i-1)H(n,i-1,m-1,j,k)\\
&+(n+r-2)^2(i-m-1)(n-i+1)(n+i-1)H(n,i-1,m,j,k)\\
&-(n+r-2)^2(i-1)^2(i-m+1)H(n,i,m-1,j,k)\\
&+(n+r-2)^2(i-1)^2(i-m-r+1)H(n,i,m,j,k)=0.
\end{aligned} \tag{8}$$

The preceding recurrence was found by `FindRecurrence`. The theorem will follow if we can establish the correct values of $h(n,1,m)$, $h(n,n,n)$, and $h(n,i,1)$.

Case $h(n,1,m)$. Maple computes $h(n,1,1)=1$, and it computes

$$H_1(n,1,m,j)=\sum_{k=0}^{j-1}H(n,1,m,j,k)=\frac{(-1)^{j+1}j\binom{n+j+r-2}{j}\binom{n}{j}}{r\binom{j+m+r-2}{r}}.$$

Now $h(n,1,m)=\sum_j H_1(n,1,m,j)$, and with $H_1(n,1,m,j)$ as input, the function `sumrecursion` gives the recurrence $(n-1)(n-2+m+r)h(n,1,m)-(n+r-1)(n-m)h(n-1,1,m)=0$, and Maple gives the initial value $h(m,1,m)=0$ for $m>1$.

Case $h(n,n,n)$. Maple computes

$$H_1(n,n,n,j)=\sum_{k=0}^{j-1}H(n,n,n,j,k)=\frac{(-1)^{n+j}j\binom{2n+r+2}{n}\binom{n+j+r-3}{j-1}\binom{n}{j}}{r\binom{n+j+r-2}{r}}.$$

Similarly to the previous case, `sumrecursion` gives the recurrence $h(n,n,n)-(n-1,n-1,n-1)=0$ and, obviously, $h(1,1,1)=1$.

Case $h(n,i,1)$. We need to do something different in this case. First, we show that our conjectured inverse is symmetric. Let

$$S(n,i,j,k)=(-1)^{i+k+1}\binom{n+i+r-2}{i}\binom{n}{i}\binom{n+k+r-2}{k}\binom{n}{k}\frac{i^2\prod_{l=0}^{r-3}i+j+l}{r\prod_{l=0}^{r-2}i+k+l},$$

so that $B_{ij}(n,r)=\sum_{k=0}^{j-1}S(n,i,j,k)$. Now `zeil` produces the recurrence

$$S(n+1,i,j,k)-S(n,i,j,k)=T(n,i,j,k+1)-T(n,i,j,k)$$

where

$$T(n,i,j,k)=\frac{-(2n+r)k^2(i+k+r-2)}{(n+r-1)^2(n-i+1)(n-k+1)}S(n,i,j,k).$$

This implies that $B_{ij}(n+1,r)-B_{ij}(n,r)=T(n,i,j,j)-T(n,i,j,0)$. Now Maple tells us that

$$T(n,i,j,j)-T(n,i,j,0)-T(n,j,i,i)+T(n,j,i,0)=0,$$

which means that $B_{ij}(n+1,r)-B_{ji}(n+1,r)=B_{ij}(n,r)-B_{ji}(n,r)$. Maple also tells us that

$$B_{in}(n,r)-B_{ni}(n,r)=\frac{\binom{n+i+r-2}{i}i(n+i+r-3)!\Gamma(2-r)\Gamma(2-n-i-r)(-1)^i}{r(n+i-1)!(i+r-2)!\Gamma(2-n-r)\Gamma(2-i-r)\Gamma(1-i)},$$

which implies $B_{in}(n,r)-B_{ni}(n,r)=0$.

Since $R_n(b_k)$ and $B(n,r)$ are symmetric, the $(1,i)$-entry of $R_n(b_k)B(n,r)$ equals the $(i,1)$-entry of $B(n,r)R_n(b_k)$. The former is $\sum_{j=1}^{n}\sum_{k=1}^{i-1}U(n,i,j,k)$, where

$$U(n,i,j,k)=\binom{j+r-1}{r}^{-1}S(n,j,i,k).$$

The function `zeil` produces

$$Y(n,i,j,k)=\frac{-(2n+r)k^2(j+k+r-2)}{(n+r-1)^2(n-j+1)(n-k+1)}U(n,i,j,k)$$

which satisfies

$$U(n+1,i,j,k)-U(n,i,j,k)=Y(n,i,j,k+1)-Y(n,i,j,k).$$

Then we have

$$\sum_{k=1}^{i-1}U(n+1,i,j,k)-\sum_{k=1}^{i-1}U(n,i,j,k)=Y(n,i,j,i)-Y(n,i,j,0),$$

and Maple tells us that $\sum_{j=1}^{n}Y(n,i,j,i)-Y(n,i,j,0)=0$. All that remains is to check the initial value $\sum_{j=1}^{i}\sum_{k=1}^{i-1}U(i,i,j,k)=0$. Maple also tells us that

$$\sum_{j=1}^{i}\sum_{k=1}^{i-1}U(i,i,j,k)=\frac{\Gamma(1-r)\Gamma(2i-r)\Gamma(2i+r+1)\Gamma(2i+r-1)}{\Gamma(-r-1)^2\Gamma(i+r+1)^2\Gamma(i+r)}\frac{(-1)^i}{(i-1)\Gamma(-i)},$$

which implies that $\sum_{j=1}^{i}\sum_{k=1}^{i-1}U(i,i,j,k)=0$ when $i>1$. □

We consider reciprocal Hankel matrices based on one more sequence of binomial coefficients. Let $c_k=\binom{k+3}{3}$.

Theorem 5.3: Let $C(n)$ be the $n\times n$ matrix whose (i,j)-entry is

$$C_{ij}(n)=\sum_{k=0}^{j-1}(-1)^{i+k+1}\binom{n+i+2}{i+k+1}\binom{n+k+1}{i+k+1}\binom{i+k+1}{i}\binom{i+k}{i}\frac{i(j-k)}{3}.$$

Then $C_{ij}(n)$ is an integer, and $C(n)$ is the inverse of the matrix $R_n(c_k)$.

Proof: First, we show that each summand of the sum that defines each entry is an integer. It is well known that, if $a\equiv 0 \pmod 3$, $b\equiv 1 \pmod 3$, and $c\equiv 2 \pmod 3$, then $\binom{a}{b}$, $\binom{a}{c}$, and $\binom{b}{c}$ are all divisible by 3. Using this fact, we find that one of the terms $\binom{i+k+1}{i}$, $\binom{i+k}{i}$, or i is divisible by 3 unless $i\equiv 1 \pmod 3$ and $k\equiv 0 \pmod 3$. But now $n+i+2\equiv n \pmod 3$, $n+k+1\equiv n+1 \pmod 3$, and $i+k+1\equiv 2 \pmod 3$. Thus, 3 divides one of the terms $\binom{n+i+2}{i+k+1}$ or $\binom{n+k+1}{i+k+1}$.

The proof that $C(n)$ is the inverse of $R_n(c_k)$ is similar to the proof of Theorem 5.2. Let

$$Z(n,i,m,j,k)=(-1)^{i+k+1}\binom{n+i+2}{i+k+1}\binom{n+k+1}{i+k+1}\binom{i+k+1}{i}\binom{i+k}{i}\frac{i(j-k)}{3\binom{j+m+2}{3}},$$

so that $z(n,i,m)=\sum_{j=1}^{n}\sum_{k=0}^{j-1}Z(n,i,m,j,k)$ is the (i,m)-entry of $C(n)R_n(c_k)$. Then Z satisfies the recurrence

$$\begin{aligned}(n-i+1)(n+i+1)(Z(n-1,i-1,m,j,k)-Z(n,i-1,m,j,k))\\ +i(i-1)(Z(n-1,i,m,j,k)-Z(n,i,m,j,k))=0.\end{aligned}$$

Now the proof proceeds similarly to the proof of Theorem 5.2, except that we do not have to do the difficult initial value $m=1$. □

One might wonder whether there is not a simpler formula than the one we give for $B_{ij}(n,r)$. If we fix i and j and consider B_{ij} as a polynomial of n, then it usually has an irreducible factor of degree $\min\{2i-2, 2j-2\}$. Thus, it seems unlikely that one could avoid the sum in the given formula. The next section suggests that the given sum is the "right" way to describe $B_{ij}(n,r)$.

6. RECIPROCAL HANKEL MATRICES BASED ON FIBONOMIAL COEFFICIENTS

Remarkably, by changing the exponent of -1 and changing the binomial coefficients to Fibonomial coefficients in the formula for B_{ij}, we get a formula for the entries of the inverses of reciprocal Hankel matrices based on Fibonomial coefficients.

Let $d_k = d_k(r)$ be the Fibonomial coefficient $\left(\binom{k+r-1}{r}\right)$.

Conjecture 6.1: Let $D(n,r)$ be the $n \times n$ matrix whose (i,j)-entry is

$$D_{ij} = D_{ij}(n,r) = \sum_{k=0}^{j-1} (-1)^{e(n,i,k)} \left(\binom{n+i+r-2}{i}\right)\left(\binom{n}{i}\right) \times \left(\binom{n+k+r-2}{k}\right)\left(\binom{n}{k}\right) \frac{F_i^2 \prod_{l=0}^{r-3} F_{i+j+l}}{F_r \prod_{l=0}^{r-2} F_{i+k+l}}.$$

Then the $D(n,r)$ is the inverse of the matrix $R_n(d_k)$.

We have verified this conjecture for $n \le 16$ and $r \le 10$. (We assume that r is a positive integer.) We also observe that the inverse of a reciprocal Hankel matrix based on Fibonomial coefficients has integer entries exactly when the corresponding reciprocal Hankel matrix based on binomial coefficients has integer entries. This may just be a consequence of known divisibility properties of the Fibonomials. It seems likely that this conjecture may be proved by combining the methods of the proofs of Theorem 2.2 and Theorem 5.2, and that it may be extended to the corresponding sequence of x-Fibonomial coefficients.

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