ON THE NUMBER OF MAXIMAL INDEPENDENT SETS OF VERTICES IN STAR-LIKE LADDERS

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1. INTRODUCTION

Let MIS stand for the *maximal independent set* of vertices. Denote the number of MIS of G by M_G. Sanders [1] exhibits a tree $p(P_n)$, called an *extended path*, formed by appending a single degree-one vertex to each vertex of a path on n vertices, and proves $M_{p(P_n)} = F_{n+2}$. In this paper we introduce a new class of graphs, called *star-like ladders*, and show that the number of MIS in star-like ladders has a connection to the Fibonacci numbers. In particular, we show that $M_{L_p} = 2F_{p+1}$, where L_p is the ladder with p squares.

Remember that the *ladder* L_p, $p \geq 1$, is the graph with $2p+2$ vertices $\{u_i, v_i \mid i = 0, 1, \ldots, p\}$ and edges $\{u_i u_{i+1}, v_i v_{i+1} \mid i = 0, 1, \ldots, p-1\} \cup \{u_i v_i \mid i = 0, 1, \ldots, p\}$. Two *end edges* of the ladder L_p are the edges joining vertices of degree 2.

The graph obtained by identifying an end edge of ladder L_p with an edge e of a graph G is denoted by $G[e, p]$. For the sake of completeness, we will put $G[e, 0] = G$. If $p_1, \ldots, p_k \in \mathbf{N}$ and $e_1, \ldots, e_k$ are the edges of G, then we will write $G[(e_1, \ldots, e_k), (p_1, \ldots, p_k)]$ for $G[e_1, p_1] \ldots [e_k, p_k]$. The *star-like ladder* $SL(p_1, \ldots, p_k)$ is the graph $K_2[(e, \ldots, e), (p_1, \ldots, p_k)]$, where e is the edge of K_2. We have that $L_p = SL(p) = K_2[e, p]$, $p \in \mathbf{N}$.

2. MIS IN GRAPHS WITH PENDANT LADDERS

Graph G has *pendant ladders* if there is a graph G^*, the edges e_i of G^* and $p_i \in \mathbf{N}$, $i = 1, \ldots, k$, $k \geq 1$, such that $G = G^*[(e_1, \ldots, e_k), (p_1, \ldots, p_k)]$. In the next lemma, we give the recurrence formula for M_G when G has pendant ladders.

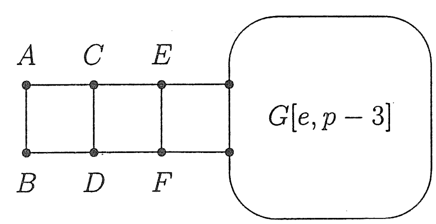

FIGURE 1. The Graph $G[e, p]$

Lemma 1: If e is an edge of a graph G and $p \in \mathbf{N}$, $p \geq 3$, then

$$M_{G[e,\,p]} = M_{G[e,\,p-1]} + M_{G[e,\,p-2]}. \tag{1}$$

Proof: Let M be MIS in $G[e, p]$. Then, for every vertex v of $G[e, p]$, either $v \in M$ or v has a neighbor in M; otherwise, $M \cup \{v\}$ is the independent set of vertices properly containing M. Further, exactly one of vertices A and B (see Fig. 1) belongs to M. Obviously, M cannot contain both A and B, but if M contains neither A nor B, then from above it must contain both C and D, which is a contradiction.

Suppose that $A \in M$. Then $M-\{A\}$ is MIS in $G[e, p-1]$ or $G[e, p-2]$, but not both. For every MIS M' in $G[e, p-1]$ containing D, we have that $M' \cup \{A\}$ is MIS in $G[e, p]$. If $D \notin M$, then $F \in M$ and $M-\{A\}$ is MIS in $G[e, p-2]$. Also, for every MIS M' in $G[e, p-2]$ containing F, we have that $M' \cup \{A\}$ is MIS in $G[e, p]$. Similar holds if $B \in M$. Since every MIS in $G[e, p-1]$ contains exactly one of C and D, and every MIS in $G[e, p-2]$ contains exactly one of E and F, we conclude that (1) holds. □

Let j_i denote the i^{th} coordinate of the vector j.

Theorem 1: If $e_1, \ldots, e_k$ are the edges of a graph G and $p_1, \ldots, p_k \in \mathbf{N} \setminus \{1, 2\}$, then

$$M_{G[(e_1, \ldots, e_k), (p_1, \ldots, p_k)]} = \sum_{j \in \{1, 2\}^k} \left(\prod_{i=1}^{k} F_{p_i - 3 + j_i} \right) M_{G[(e_1, \ldots, e_k), j]}. \tag{2}$$

Proof: First we prove (2) for $k=1$ by induction on p_1. If $p_1 = 3$, then

$$M_{G[e_1, 3]} = F_2 M_{G[e_1, 2]} + F_1 M_{G[e_1, 1]}.$$

Supposing that (2) is true for $k=1$ and all $p_1 < p$ for some p, we have that

$$\begin{aligned} M_{G[e_1, p]} &= M_{G[e_1, p-1]} + M_{G[e_1, p-2]} \\ &= (F_{p-2} M_{G[e_1, 2]} + F_{p-3} M_{G[e_1, 1]}) + (F_{p-3} M_{G[e_1, 2]} + F_{p-4} M_{G[e_1, 1]}) \\ &= F_{p-1} M_{G[e_1, 2]} + F_{p-2} M_{G[e_1, 1]}. \end{aligned}$$

Now we prove (2) by induction on k. Suppose that (2) is true for some $k=n$ and for all $p_1, \ldots, p_n \in \mathbf{N} \setminus \{1, 2\}$. Let $p = (p_1, \ldots, p_n, p_{n+1})$, $p' = (p_1, \ldots, p_n)$, and $e = (e_1, \ldots, e_n, e_{n+1})$, $e' = (e_1, \ldots, e_n)$. We have that

$$\begin{aligned} M_{G[e, p]} = M_{G[(e', p')[e_{n+1}, p_{n+1}]} &= \sum_{j \in \{1, 2\}^n} \left(\prod_{i=1}^{n} F_{p_i - 3 + j_i} \right) M_{G[e', j][e_{n+1}, p_{n+1}]} \\ &= \sum_{j \in \{1, 2\}^n} \left(\prod_{i=1}^{n} F_{p_i - 3 + j_i} \right) (F_{p_{n+1}-1} M_{G[e', j][e_{n+1}, 2]} + F_{p_{n+1}-2} M_{G[e', j][e_{n+1}, 1]}) \\ &= \sum_{j \in \{1, 2\}^{n+1}} \left(\prod_{i=1}^{n+1} F_{p_i - 3 + j_i} \right) M_{G[e, j]}. \quad \square \end{aligned}$$

If we define $F_0 = F_2 - F_1 = 0$ and $F_{-1} = F_1 - F_0 = 1$, then we can drop the assumption that $p_i \neq 1, 2$, $i = 1, \ldots, k$ in the previous theorem.

3. MIS IN STAR-LIKE LADDERS

Theorem 2: If $p_1, \ldots, p_k \in \mathbf{N}$, then

$$M_{SL(p_1, \ldots, p_k)} = (2^k - 2) \prod_{i=1}^{k} F_{p_i} + 2 \prod_{i=1}^{k} F_{p_i + 1}.$$

Proof: Let $j \in \{1, 2\}^k$ with $j_{(1)}$ coordinates equal to 1, and $j_{(2)}$ coordinates equal to 2. We prove that

$$M_{K_2[(e,\ldots,e),\, j]} = 2^k + 2 \cdot 2^{j(2)} - 2, \tag{3}$$

where e is the edge of K_2. Let M be MIS of $K_2[(e, \ldots, e), j]$ (see Fig. 2). If $X \in M$, then $A_i \in M$ for $i = 1, \ldots, j_{(1)}$, and either $C_i \in M$ or $D_i, E_i \in M$ for $i = 1, \ldots, j_{(2)}$. Similar holds if $Y \in M$, and this gives $2 \cdot 2^{j(2)}$ MIS of $K_2[(e, \ldots, e), j]$. If $X, Y \notin M$, then either $A_i \in M$ or $B_i \in M$ for $i = 1, \ldots, j_{(1)}$ and either $C_i, F_i \in M$ or $D_i, E_i \in M$ for $i = 1, \ldots, j_{(2)}$, giving 2^k possibilities. Here we must exclude sets $\{A_1, \ldots, A_{j(1)}, D_1, E_1, \ldots, D_{j(2)}, E_{j(2)}\}$ and $\{B_1, \ldots, B_{j(1)}, C_1, F_1, \ldots, C_{j(2)}, F_{j(2)}\}$ which are not MIS, and so it follows that (3) holds. Now

$$\begin{aligned} M_{SL(p_1,\ldots,p_k)} &= \sum_{j \in \{1,2\}^k} \left(\prod_{i=1}^{k} F_{p_i - 3 + j_i} \right) M_{K_2[(e,\ldots,e),\, j]} \\ &= \sum_{j \in \{1,2\}^k} \left(\prod_{i=1}^{k} F_{p_i - 3 + j_i} \right) (2^k + 2 \cdot 2^{j(2)} - 2) \\ &= (2^k - 2) \prod_{i=1}^{k} (F_{p_i - 2} + F_{p_i - 1}) + 2 \prod_{i=1}^{k} (F_{p_i - 2} + 2F_{p_i - 1}) \\ &= (2^k - 2) \prod_{i=1}^{k} F_{p_i} + 2 \prod_{i=1}^{k} F_{p_i + 1}. \quad \square \end{aligned}$$

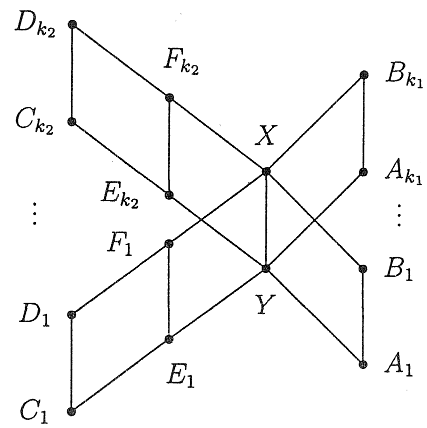

FIGURE 2. The Graph $K_2[(e, \ldots, e), (1, \ldots, 1, 2, \ldots, 2)]$

As an immediate consequence, we get

Corollary 1: If $p \in \mathbf{N}$, then $M_{L_p} \doteq 2F_{p+1}$.

REFERENCE

1. L. K. Sanders. "A Proof from Graph Theory for a Fibonacci Identity." *The Fibonacci Quarterly* **28.1** (1990):48-55.

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