ON THE INTEGER SOLUTION OF THE EQUATION $5x^2 \pm 6x + 1 = y^2$ AND SOME RELATED OBSERVATIONS

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The integer solution of the equation

$$5x^2 \pm 6x + 1 = y^2 \tag{1}$$

is interesting because of the Fibonacci and Lucas relationships that appear.

One method of solving the problem involves the solution of the Pythagorean, (Py), equation

$$X^2 + Y^2 = Z^2 , \tag{2}$$

where $X = 2ab$, $Y = a^2 - b^2$, $Z = a^2 + b^2$, and $a > b$. Since no other restrictions are placed on a and b this solution of (2) is not necessarily primitive.

When $4x^2$ is added to both sides of (1) we obtain

$$9x^2 \pm 6x + 1 = y^2 + 4x^2 \quad \text{or} \tag{3}$$

$$(3x \pm 1)^2 = y^2 + (2x)^2 . \quad \text{Now let} \tag{4}$$

$$3x \pm 1 = Z = a^2 + b^2 , \tag{5a}$$

$$y = Y = a^2 - b^2 \quad \text{and} \tag{5b}$$

$$2x = X = 2ab \quad \text{or} \tag{5c}$$

$$x = ab. \tag{5d}$$

Substituting this value of x in (5a) we get

$$3ab \pm 1 = a^2 + b^2 \quad \text{or}$$

$$a^2 - 3ab + (b^2 \pm 1) = 0 . \tag{6}$$

Solving this equation for $a > b$ we have

$$a = \frac{3b + \sqrt{9b^2 - 4(b^2 \pm 1)}}{2} = \frac{3b + \sqrt{5b^2 \pm 4}}{2} \tag{7}$$

If the values of b are such that $5b^2 \pm 4 = \square$ then $3b + \sqrt{5b^2 \pm 4}$ is always even and therefore a is always integral. Changing equation (7) to

(8) $$2a = 3b + \sqrt{5b^2 \pm 4}$$

we prepare Table I by filling in the column under b with the Fibonacci numbers, F, and the column beneath the radical sign with the Lucas numbers, L. The rest of the table is then calculated.

Table I Showing Fibonacci and Lucas Relationships Involved in the Solution of
$$5x^2 \pm 6x + 1 = y^2$$

n,	a,	b;	$2a = 3b + \sqrt{5b^2 \pm 4}$;	$x = ab$;	$y = a^2 - b^2$
0,	1,	0;	2 = 0 + 2;	0;	1
1,	2,	1;	4 = 3 + 1;	2;	3
2,	3,	1;	6 = 3 + 3;	3;	8
3,	5,	2;	10 = 6 + 4;	10;	21
4,	8,	3;	16 = 9 + 7;	24;	55
5,	13,	5;	26 = 15 + 11;	65;	144
6,	21,	8;	42 = 24 + 18;	168;	377
-	-	-	- - -	-	-

$$n,\ F_{n+2},\ F_n;\ 2F_{n+2} = 3F_n + L_n;\ F_{n+1}^2 - (-1)^n = F_nF_{n+2};\ F_{n+2}^2 - F_n^2 = F_{2n+2}$$
$$= (L_{n+1}^2 - F_{n+1}^2)/4;\ = L_{n+1}F_{n+1}$$

Note that $x_n + x_{n+1} = F_{2n+3} = F_{n+1}^2 + F_{n+2}^2$

The solution to equation (1) is

(9) $$x_n = F_{n+1}^2 - (-1)^n = F_nF_{n+2} = 2(x_{n-1} + x_{n-2}) - x_{n-3}$$

(10) $$y_n = F_{n+2}^2 - F_n^2 = F_{2n+2} = L_{n+1}F_{n+1} = 3y_{n-1} - y_{n-2}\ .$$

From (9) we have the interesting recurrent equation

$$x_n = 2(x_{n-1} + x_{n-2}) - x_{n-3}\ ,$$

which can be expressed with Fibonacci terms as:

$$(11)\qquad F_{n+1}^2-(-1)^n = 2\left[F_n^2-(-1)^{n-1}+F_{n-1}^2-(-1)^{n-2}\right] - \left[F_{n-3}^2-(-1)^{n-3}\right] .$$

The (-1) terms disappear so that

$$(12)\qquad F_{n+1}^2 = 2F_n^2 + 2F_{n-1}^2 - F_{n-2}^2 \qquad \text{or}$$

$$F_{n+1}^2 - F_{n-1}^2 = (F_n^2 - F_{n-2}^2) + (F_n^2 + F_{n-1}^2) \qquad \text{or}$$

$$F_{2n} = F_{2n-2} + F_{2n-1} \qquad \text{or}$$

$$F_{2n} = F_{2n}$$

and thus we have proved (11) and (12). Equation (12) can be written as

$$2(F_n^2+F_{n-1}^2) = F_{n+1}^2+F_{n-2}^2 \qquad \text{or}$$

$$(13)\qquad 2F_{2n-1} = F_{n+1}^2+F_{n-2}^2 \quad ,$$

an interesting Fibonacci identity. Another interesting Fibonacci identity turns up when the appropriate F and L terms are substituted in equation (8), $2a = 3b + \sqrt{5b^2 \pm 4}$. We have

$$(14)\qquad 2F_{n+2} = 3F_n+L_n \quad .$$

This identity is proved by adding $3F_n$ to each side of the identity $F_{n-1}+F_{n+1} = L_n$ as follows:

$$F_{n-1}+F_{n+1} = L_n$$

$$F_n+F_n+F_n = 3F_n$$

$$F_n+(F_n+F_{n-1})+(F_{n+1}+F_n) = 3F_n+L_n$$

$$(F_n+F_{n+1})+F_{n+2} = 3F_n+L_n$$

$$F_{n+2}+F_{n+2} = 3F_n+L_n$$

$$2F_{n+2} = 3F_n+L_n$$

Equation (1) can be written as

$$5x_n^2 + 6(-1)^n x_n + 1 = y_n^2 \tag{14}$$

and when the appropriate F terms are substituted, this equation becomes

$$5F_n^2 F_{n+2}^2 + 6(-1)^n F_n F_{n+2} + 1 = F_{2n+2}^2 \tag{15}$$

which equation is equivalent to

$$5F_{n-1}^2 F_{n+1}^2 - 6(-1)^n F_{n-1} F_{n+1} + 1 = F_{2n}^2 \quad \text{or} \tag{16}$$

$$5[F_n^2 + (-1)^n]^2 - 6(-1)^n [F_n^2 + (-1)^n] + 1 = F_{2n}^2 = L_n^2 F_n^2 \; . \tag{17}$$

When the indicated operations are performed we have successively

$$5[F_n^4 + 2(-1)^n F_n^2 + (-1)^{2n}] - 6(-1)^n [F_n^2 + (-1)^n] + 1 = L_n^2 F_n^2$$

$$5F_n^4 + 10(-1)^n F_n^2 + 5 - 6(-1)^n F_n^2 - 6(-1)^{2n} + 1 = L_n^2 F_n^2$$

$$5F_n^4 + 4(-1)^n F_n^2 = L_n^2 F_n^2$$

$$5F_n^2 + 4(-1)^n = L_n^2$$

and thus we have proved the identities (16) and (17).

Now we examine the solution of equation (2), $X^2 + Y^2 = Z^2$, where F and L terms are used for (a) and (b). For this purpose we first prepare Table II where the a's and b's are transferred from Table I. The rest of Table II is then calculated.

The solution of $X^2 + Y^2 = Z^2$ is

$$X = 2ab = 2F_n F_{n+2} \tag{18a}$$

$$Y = a^2 - b^2 = F_{n+2}^2 - F_n^2 = F_{2n+2}$$

$$Z = a^2 + b^2 = F_{n+2}^2 + F_n^2 = 3F_{n+1}^2 - 2(-1)^n \; .$$

Table II Showing Fibonacci and Lucas Relationships Involved in the Solution of $X^2+Y^2 = Z^2$

$X = 2ab$, $Y = a^2-b^2$, $Z = a^2+b^2$, $a = F_{n+2}$, and $b = F_n$.

n,	a,	b,	2ab=X,	$a^2-b^2=Y$,	$a^2+b^2=Z$,	a-b,	a+b
0,	1	0	1	1	1	1	1
1,	2	1	4	3	5	1	3
2,	3	1	6	8	10	2	4
3,	5	2	20	21	29	3	7
4,	8	3	48	55	73	5	11
5,	13	5	130	144	194	8	18
6,	21	8	336	377	505	13	29
-	-	-	-	-	-	-	-
n,	F_{n+2},	F_n,	$2F_nF_{n+2}$,	$F_n^2-F_{n+2}^2$,	$F_n^2+F_{n+2}^2$,	F_{n+1},	L_{n+1}
			$2[F_{n+1}^2-(-1)^n]$,	F_{2n+2},	$(L_{n+1}^2+F_{n+1}^2)/2$,		
			$(L_{n+1}^2-F_{n+1}^2)/2$,	$L_{n+1}F_{n+1}$,	$3F_{n+1}^2-2(-1)^n$		

The identity (18c), $F_{n+2}^2+F_n^2 = 3F_{n+1}^2 - 2(-1)^n$, is equivalent to

(19) $F_{n+1}^2+F_{n-1}^2 = 3F_n^2+2(-1)^n$ but

$F_{n+1}F_{n-1} = F_n^2+(-1)^n$ and

$2F_{n+1}F_{n-1} = 2F_n^2+2(-1)^n$ and therefore

$F_{n+1}^2+F_{n-1}^2 = 2F_{n+1}F_{n-1} + F_n^2$ or

$F_{n+1}^2-2F_{n+1}F_{n-1}+F_{n-1}^2 = F_n^2$ or

$(F_{n+1}-F_{n-1})^2 = F_n^2$ or

$F_{n+1}-F_{n-1} = F_n$ or

$F_{n+1} = F_n+F_{n-1}$

and thus we have proved the Fibonacci identity for Z in (18c).

An equivalent equation for $X^2 + Y^2 = Z^2$ is the following Fibonacci identity:

(20) $$4\left[F_{n+1}^2 - (-1)^n\right]^2 + F_{2n+2}^2 = \left[3F_{n+1}^2 - 2(-1)^n\right]^2 .$$

When the indicated operations are performed and the terms are collected this equation becomes

$$5F_{n+1}^4 - 4(-1)^n F_{n+1}^2 = F_{2n+2}^2 \qquad \text{or}$$

$$5F_n^4 + 4(-1)^n F_n^2 = F_{2n}^2 = L_n^2 F_n^2 \qquad \text{or}$$

$$5F_n^2 + 4(-1)^n = L_n^2$$

and thus we have proved the Fibonacci identity expressed by equation (20).

The following equations represent further observations.

(21) $$Z + X = a^2 + 2ab + b^2 = (a+b)^2 = (F_{n+2} + F_n)^2 = L_{n+1}^2 \qquad \text{and}$$

(22) $$Z - X = a^2 - 2ab + b^2 = (a-b)^2 = (F_{n+2} - F_n)^2 = F_{n+1}^2 .$$

Adding these equations and dividing by 2 we have

(23) $$Z = (L_{n+1}^2 + F_{n+1}^2)/2$$

and subtracting the equations and dividing by 2 we get

(24) $$X = (L_{n+1}^2 - F_{n+1}^2)/2$$

and multiplying (21) by (22) we obtain

(25) $$Z^2 - X^2 = L_{n+1}^2 F_{n+1}^2 = Y^2 \qquad \text{or}$$

(26) $$Y = L_{n+1} F_{n+1} .$$

The area, A, of the Py triangle is

(27) $$A = ab(a^2 - b^2) .$$

In general, the module, $ab(a^2 - b^2)$, is divisible by 6, consequently when the appropriate F and/or L terms are substituted in the module the resulting expression must likewise be divisible by 6. Thus the following expressions are all divisible by 6:

$$F_n F_{n+2}(F_{n+2}^2 - F_n^2) \; ; \quad [F_{n+1}^2 - (-1)^n][F_{n+2}^2 - F_n^2] \; ;$$

$$F_n F_{n+1} F_{n+2} L_{n+1} \; ; \quad F_n F_{n+2} F_{2n+2} \; ;$$

$$[F_{n+1}^2 - (-1)^n] F_{2n+2} \; ; \quad [F_{n+1}^2 - (-1)^n] F_{n+1} L_{n+1} \; ,$$

and

$$(L_{n+1}^2 - F_{n+1}^2) L_{n+1} F_{n+1} \qquad \text{or}$$

$$(L_{n+1}^2 - F_{n+1}^2) F_{2n+2} \qquad \text{or}$$

$$(L_{n+1}^2 - F_{n+1}^2)(F_{n+2}^2 - F_n^2)$$

are all divisible by 24 since $ab = (L_{n+1}^2 - F_{n+1}^2)/4$.

In the foregoing considerations the values of a and b were restricted by equation (1) to $a = F_{n+2}$, $b = F_n$. If now, in the solution of a Py triangle, we substitute for a and b any arbitrary F and/or L terms then Fibonacci and/or Lucas number identities are easily produced in infinite variety and divisibility expressions are easily produced and proved.

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