# A DIVISIBILITY PROPERTY OF FIBONACCI NUMBERS

Lenard Weinstein
Massachusetts Institute of Technology

The following is an interesting theorem concerning the first 2n Fibonacci numbers:

Theorem

Give any set of $n + 1$ Fibonacci numbers selected from the set $F_1, F_2, \ldots, F_{2n}$, it is always possible to choose two elements from the $n + 1$ Fibonacci numbers such that one divides the other.

This theorem will be proved using the following two theorems:

Theorem 1:

Give any set of $n + 1$ integers selected from the set 1, 2, ..., 2n, it is always possible to choose two elements from the $n + 1$ integers such that one divides the other.

Proof:

We shall use induction. The theorem is trivially true for the case $n = 1$. Assume it true for $n = k$. If $n = k+1$, we must prove that any set of k+2 integers selected from the set 1, 2, ..., 2(k+1) contains two elements such that one divides the other. If the k+2 integers are contained in the set 1, 2, ..., 2k, then by the inductive hypothesis, the theorem is true. Similarly, if k+1 of the integers are contained in the set 1, 2, ..., 2k, and the other integer is either 2k+1 or 2(k+1), the same reasoning applies as above. If only k of the integers belong to the set 1, 2, ..., 2k, then the other two integers must be the numbers 2k+1 and 2(k+1). Consider any set of k numbers from the set 1, 2, ..., 2k and the integer k+1. By the inductive hypothesis, we can find two numbers from this set such that one number divides the other. Since k+1 does not divide any number but itself, in the set 1, 2, ..., 2k, at best it is divisible by another element of the set. Any number that divides k+1, though, divides 2(k+1). Thus, any set of k elements chosen from the set $\{1, 2, \ldots, 2k\}$ and the element 2(k+1), contains two numbers such that one number divides the other. Therefore, the set of any k integers from the set 1, 2, ..., 2k , plus the numbers 2k+1 and 2(k+1), also contains two elements such that one divides the other.

Theorem 2:

If $F_n$ is the $n^{th}$ Fibonacci number ($F_1 = 1$, $F_2 = 1$), then $F_{(a,b)} = (F_a, F_b)$ [1] . ( (a, b) is the greatest common divisor of the integers a and b. ) This is a widely known theorem, easily proved.

It follows that if any set of n+1 Fibonacci numbers $F_{a_1}$, $F_{a_2}$, ... $F_{a_{n+1}}$, is chosen from the set $F_1, F_2, \ldots, F_{2n}$, there exist two elements of the n+1 Fibonacci numbers such that one number divides the other. For consider the numbers $a_1, a_2, \ldots, a_{n+1}$. By theorem 1, we can find two numbers $a_j$, $a_k$ from these n+1 integers such that $a_j | a_k$. Thus $(a_j, a_k) = a_j$. It follows that $F_{(a_j, a_k)} = F_{a_j} = (F_{a_j}, F_{a_k})$ by theorem 2. This means $F_{a_j} | F_{a_k}$, and the theorem is proved.

## REFERENCES

1. N. N. Vorob'ev, Fibonacci Numbers, Blaisdell, New York, 1961, p. 30.

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