ADVANCED PROBLEMS AND SOLUTIONS

Edited by V.E. Hoggatt, Jr., San Jose State College, San Jose, Calif.

Send all communications concerning Advanced Problems and Solutions to Verner E. Hoggatt, Jr., Mathematics Department, San Jose State College, San Jose, California. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within three months after publication of the problems.

H-89 *Proposed by Maxey Brooke, Sweeny, Texas.*

Fibonacci started out with a pair of rabbits, a male and a female. A female will begin bearing after two months and will bear monthly thereafter. The first litter a female bears is twin males, thereafter she alternately bears female and male.

Find a recurrence relation for the number of males and females born at the end of the n^{th} month and the total rabbit population at that time.

H-90 *Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, Calif.*

Let the total population after n time periods be the sequence $\{F_n^3\}_{n=2}^{\infty}$ determine the common birth sequence for every female rabbit and tie it in with the value of the Fibonacci polynomials at $x = 2$. ($f_0(x) = 0$, $f_1(x) = 1$ and $f_{n+2}(x) = xf_{n+1}(x) + f_n(x)$; $n \geq 0$).

H-91 *Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.*

Let $m = \left[\frac{k}{2}\right]$, then show

$$F_{kn}/F_n = \sum_{j=0}^{m=1} (-1)^{jn} L_{k-1-2j} + e_n \quad ,$$

where

$$e_n = \begin{cases} (-1)^{mn} & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even} \end{cases}$$

and $[x]$ is the greatest integer not exceeding x.

H-92 *Proposed by Brother U. Alfred, St. Mary's College, Calif.*

Prove or Disprove: A part from F_1, F_2, F_3, F_4, no Fibonacci number, F_i $(i > 0)$ is a divisor of a Lucas Number.

H-93 *Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.*

Show that

$$F_n = \sum_{k=1}^{\overline{n-1}} \left(3 + 2\cos\frac{2k\pi}{n}\right)$$

$$L_n = \sum_{k=1}^{\overline{n-2}} \left(3 + 2\cos\frac{(2k+1)\pi}{n}\right) \qquad ,$$

where $\bar{n}$ is the greatest integer contained in $n/2$.

SOLUTIONS

H-50 *Proposed by Ralph Greenberg, Philadelphia, Pa. and H. Winthrop, University of So. Florida, Tampa, Fla.*

Show

$$\sum_{n_1+n_2+n_3+\cdots+n_i=n} \Pi n_i = F_{2n}$$

where the sum is taken over all partitions of n into positive integers and the order of distinct summands is considered.

A paper by D. A. Lind and V. E. Hoggatt, Jr., "Composition Formulas Derived from Birth Sequences," will appear soon in the Fibonacci Quarterly, and will discuss this among many other examples.

H-22 *Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, Calif.*

If

$$P(x) = \prod_{i=1}^{\infty}(1 + x^{F_i}) = \sum_{n=0}^{\infty} R(n)x^n \quad ,$$

then show

$$\text{(i)} \qquad R(F_{2n} - 1) = n$$
$$\text{(ii)} \qquad R(N) > n \text{ if } N > F_{2n} - 1 \quad .$$

H-53 *Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, Calif. and S. L. Basin, Sylvania Electronics Systems, Mt. View, Calif.*

The Lucas sequence $L_1 = 1$, $L_2 = 3$; $L_{n+2} = L_{n+1} + L_n$ for $n \quad 1$ is incomplete (see V. E. Hoggatt, Jr. and C. King, Problem E-1424 American Monthly, Vol. 67, No. 6, June-July 1960, p. 593, since every integer n is not the sum of distinct Lucas numbers. OBSERVE THAT 2, 6, 9, 13, 17, ··· cannot be so represented. Let $M(n)$ be the number of positive integers less than n which cannot be so represented. Show

$$M(L_n) = F_{n-1} \quad .$$

Find, if possible, a closed form solution for $M(n)$.

A paper by David Klarner to appear soon in the Fibonacci Quarterly completely answers these questions.

H-55 *Proposed by Raymond Whitney, Lock Haven State College, Lock Haven, Pa.*

Let F(n) and L(n) denote the n^{th} Fibonacci and n^{th} Lucas numbers, respectively.

Given $U(n) = F(F(n))$, $V(n) = F(L(n))$, $W(n) = L(L(n))$ and $X(n) = L(F_n)$, find recurrence relations for the sequences U(n), V(n), W(n), and X(n).

A paper by student Gary Ford to appear soon in the Fibonacci Quarterly offers several answers to this problem. Also a paper by R. Whitney deals with this and will appear shortly.

H-52 *Proposed by Brother U. Alfred, St. Mary's College, Calif.*

Prove that the value of the determinant

$$\begin{vmatrix} u_n^2 & u_{n+2}^2 & u_{n+4}^2 \\ u_{n+2}^2 & u_{n+4}^2 & u_{n+6}^2 \\ u_{n+4}^2 & u_{n+6}^2 & u_{n+8}^2 \end{vmatrix}$$

is $18(-1)^{n+1}$.

Solution by V. E. Hoggatt, Jr., San Jose State College, San Jose, Calif.

Since $F_{2n}^2 = (L_{4n} - 2)/5$, the auxiliary polynomial satisfied by F_{2n}^2 is the product of the auxiliary polynomials for L_{4n}: $(x^2 - 7x + 1)$ and for $C_n = -2$: $(x - 1)$ or

$$(x^2 - 7x + 1)(x - 1) = x^3 - 8x^2 + 8x - 1 .$$

Therefore the recurrence relation for F_{2n}^2 is

$$u_{n+6}^2 = 8u_{n+4}^2 - 8u_{n+2}^2 + u_n^2 .$$

Thus $D_{n+1} = (-1)D_n$ by using the above recurrence relation after multiplying the first column of D_n by -1. The value of D_0 is -18, therefore $D_n = 18(-1)^{n+1}$.

TWO BEAUTIES

H-47 *Proposed by L. Carlitz, Duke University, Durham, N.C.*

Show that

$$\sum_{n=0}^{\infty} \binom{n+k-1}{n} L_n x^n = \frac{\psi_k(x)}{(1 - x - x^2)k},$$

where

$$\psi_k(x) = \sum_{r=0}^{k} (-1)^r \binom{k}{r} L_r x^r .$$

H-51 ***Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, Calif. and L. Carlitz, Duke University, Durham, N.C.***

Show that if

$$\text{(i)} \qquad \frac{xt}{1 - (2 - x)t + (1 - x - x^2)t^2} = \sum_{k=1}^{\infty} Q_k(x)t^k$$

and

$$\text{(ii)} \qquad \sum_{n=0}^{\infty} \binom{n+k-1}{n} F_n x^n = \frac{\phi_k(x)}{(1 - x - x^2)^k}$$

that

$$\phi_k(x) = \sum_{r=0}^{k} (-1)^{r+1} \binom{k}{r} F_r x^r = Q_k(x)$$

Solutions by Kathleen Weland, Gary Ford, and Douglas Lind, Undergraduate Research Program, University of Santa Clara, Santa Clara, Calif.

It is familiar that

$$(1-x)^{-k} = \sum_{n=0}^{\infty} \binom{n+k-1}{n} x^n \quad .$$

Let W_n obey $W_{n+2} = pW_{n+1} - qW_n$, $p^2 - 4q \neq 0$, and let $a \neq b$ both satisfy $x^2 - px + q = 0$, so that $a + b = p$, $ab = q$. Then $W_n = Aa^n + Bb^n$ for some constants A and B and all n. It follows that

$$\sum_{n=0}^{\infty} \binom{n+k-1}{n} W_n x^n = A(1-ax)^{-k} + B(1-bx)^{-k}$$

$$= \frac{A(1-bx)^k + B(1-ax)^k}{(1-px+qx^2)^k}$$

$$= \frac{\left[A\sum_{j=0}^{k}(-1)^j\binom{k}{j} b^j x^j + B\sum_{j=0}^{k}(-1)^j\binom{k}{j} a^j\right]}{(1-px+qx^2)^k}$$

$$= \frac{\left[\sum_{j=0}^{k}(-1)^j\binom{k}{j}(Aa^{-j} + Bb^{-j})(abx)^j\right]}{(1-px+qx^2)^k}$$

$$= \frac{\left[\sum_{j=0}^{k}(-q)^j\binom{k}{j} W_{-j}x^j\right]}{(1-px+qx^2)^k} = \frac{R_k(x)}{(1-px+qx^2)^k}$$

To get (ii) of H-51 we put $p = 1, q = -1$, $W_n = F_n$, and recalling $F_{-n} = (-1)^{n+1}F_n$ we have

$$\sum_{n=0}^{\infty} \binom{n+k-1}{n} F_n x^n = \left[\sum_{j=0}^{k}(-1)^{j+1}\binom{k}{j} F_j x^j\right] (1-x-x^2)^k \quad .$$

To obtain H-47, we set $p = 1$, $q = -1$, $W_n = L_n$, and remembering that $L_{-n} = (-1)^n L_n$ we find

$$\sum_{n=0}^{\infty} \binom{n+k-1}{n} L_n x^n = \left[\sum_{j=0}^{k} (-1)^j \binom{k}{j} L_j x^j\right] \qquad (1 - x - x^2)^k$$

We now generalize (i) of H-51. Since

$$\begin{aligned} R_k(x) &= \sum_{j=0}^{k} (-q)^j \binom{k}{j} W_{-j} x^j \\ &= A(1 - bx)^k + B(1 - ax)^k \quad , \end{aligned}$$

we have

$$\begin{aligned} \sum_{k=0}^{\infty} R_k(x)t^k &= A\sum_{k=0}^{\infty} [(1 - bx)t]^k + B\sum_{k=0}^{\infty} [(1 - ax)t]^k \\ &= \frac{A}{1 - (1 - bx)t} + \frac{B}{1 - (1 - ax)t} \\ &= \frac{A + B - (A + B)t + (Aa + Bb)x^t}{1 - (2 - px)t + (1 - px + qx^2)t^2} \end{aligned}$$

Now $W_0 = A + B$, $W_1 = Aa + Bb$, so we may write

$$(*) \qquad \sum_{k=0}^{\infty} R_k(x)t^k = \frac{W_0 + (xW_1 - W_0)t}{1 - (2 - px)\,t + (1 - px + qx^2)t^2} \quad ,$$

Putting $p = 1$, $q = -1$, $W_n = F_n$ makes $R_k(x) = Q_k(x)$ of Problem H-51, and (*) becomes

$$\sum_{k=0}^{\infty} Q_k(x)t^k = \frac{xt}{1 - (2 - x)t + (1 - x - x^2)t^2}$$

the required result.

Also solved by the proposers.

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LATE PROBLEM ADDITIONS

A SIMPLE PROOF, PLEASE!

H-94 Submitted by Robert W. Floyd, Carnegie Institute of Technology, and Donald E. Knuth, California Institute of Technology.

Let α be any irrational number, and let the notation $\{x\}$ stand for the fractional part of x. Suppose a man has accurately marked off the points 1, 0, $\{\alpha\}$, $\{2\alpha\}, \cdots$, $\{(n-1)\alpha\}$ on a line, $n \geq 1$. These $n + 1$ points divide the line segment between 0 and 1 into n disjoint intervals. Show that when the man adds the next point $\{n\alpha\}$, it falls in the largest of these n intervals; if there are several intervals which have the maximum length, the point $\{n\alpha\}$ falls in one of these maximal intervals. Furthermore, if α is the "golden ratio" $\phi^{-1} = \frac{1}{2}(\sqrt{5} - 1) = 0.618 \cdots$, then the point $\{n\alpha\}$ always divides the corresponding interval into two intervals whose lengths are in the golden ratio. A number α has the property that $\{n\alpha\}$ always divides its interval into two parts, such that the ratio of longer to shorter is less than 2, if and only if $\{\alpha\} = \phi^{-1}$ or ϕ^{-2}. (Note: The fact that the fractional parts $\{n\alpha\}$ are asymptotically equidistributed in (0, 1) is well known; this problem shows the mechanism behind that theorem, since $\{n\alpha\}$ always chooses the largest remaining open place. Furthermore, the sequence $\{n\phi\}$ is the "most equidistributed" of all these sequences.)

H-95 Proposed by J. A. H. Hunter, Toronto, Canada.

Show

$$F^3_{n+k} + (-1)^k F^3_{n-k} = L_k[F_k^2 F_{3n} + (-1)^k F_n^3].$$

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