AN EXPRESSION FOR GENERALIZED FIBONACCI NUMBERS

David E. Ferguson, Programmatics Incorporated, Los Angeles, Calif.

An interesting expression for the Fibonacci numbers is presented here which relies on the modulo three value of the subscript.

(1a) $$F_{3a} = 1 - \sum_{i=0}^{a-1} \binom{2i + a - 1}{3i} (-1)^{a-i} 8^i \qquad (a > 0)$$

(1b) $$F_{3a+1} = 1 - 2\sum_{i=0}^{a-1} \binom{2i + a}{3i + 1} (-1)^{a-i} 8^i \qquad (a \geq 0)$$

(1c) $$F_{3a+2} = 1 - 4\sum_{i=0}^{a-1} \binom{2i + a + 1}{3i + 2} (-1)^{a-i} 8^i \qquad (a \geq 0)$$

This is a special case of a more general expression for the generalized Fibonacci numbers [1].

(2a) $$V_{n,m} = 1 \qquad (m = -n + 1, \cdots, 0)$$

(2b) $$V_{n,m} = \sum_{k=1}^{n} V_{n,m-k} \quad .$$

It is seen that $F_m = V_{2,m-2}$.

It is interesting to note that these numbers arise in the analysis of polyphase merge-sorting with $n + 1$ tapes. The $V_{n,m}$ represent the total number of strings on all the tapes and also the length of strings (assuming initial length of 1) at each step of the polyphase merge process. A description of the polyphase merge-sort can be found in [2].

The general expression can be written as:

(3) $$V_{n,a(n+1)+b} = 1 + 2^{b-1}(n - 1)\sum_{i=0}^{a} \binom{in + a + b - 1}{a - i} (-1)^{a-i} (2^{n+1})^i$$

$$(b = 1, \cdots, n + 1), \quad (a \geq 0) .$$

Let

$$f_n(x) = \sum_{m=0}^{\infty} V_{n,m} x^m$$

It follows immediately that

$$(1 - x - x^2 - \cdots - x^n)f_n(x) = 1 + (n-1)x + (n-2)x^2 + \cdots + x^{n-1} .$$

Therefore

$$f_n(x) = \frac{1 + (n-1)x + (n-2)x^2 + \cdots + x^{n-1}}{1 - x - x^2 - \cdots - x^n}$$

(4)
$$= \frac{1 + (n-2)x - x^2 - \cdots - x^n}{1 - 2x + x^{n+1}}$$

$$= \frac{1}{1-x} + \frac{(n-1)x}{1 - 2x + x^{n+1}}$$

If

$$\frac{1}{1 - 2x + x^{n+1}} = \sum_{m=0}^{\infty} W_{n,m} x^m$$

the sequence $W_{n,m}$ is defined by:

(5a) $W_{n,m} = 0 \qquad (m < 0)$

(5b) $W_{n,m} = 1 \qquad (m = 0)$

(5c) $W_{n,m} = 2W_{n,m-1} - W_{n,m-n-1} \qquad (m > 0)$

From Eq. (4) $V_{n,m} = 1 + (n-1)W_{n,m-1} \qquad (m > 0)$

Theorem:

(6)
$$W_{n,a(n+1)+b} = \sum_{j=0}^{\infty} \binom{a + b + nj}{(n+1)j + b} 2^{b+(n+1)j} (-1)^{a-j}$$

(This formula immediately yields the identity (3).)

Proof:

$$\frac{1}{1 - 2x + x^{n+1}} = \sum_{m=0}^{\infty} (2x - x^{n+1})^m$$

$$= \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \binom{m}{k} 2^k (-1)^{m-k} x^{(n+1)m-nk}$$

Rearranging this sum in terms of powers of x, let $(n+1)a + b = (n+1)m - nk$. It follows that $k \equiv b \pmod{(n+1)}$, so $k = (n+1)j + b$ for some $j \geq 0$. Changing the sum on m and k into a sum on a, b and j, and noting that $m = a + b + nj$, results in:

$$\sum_{a=0}^{\infty} \sum_{b=0}^{\infty} x^{(n+1)a+b} \sum_{j=0}^{\infty} \binom{a+b+nj}{(n+1)j+b} 2^{(n+1)j+b} (-1)^{a-j}$$

This completes the proof. A similar method was used by Polya [3] to solve another recurrence relation.

Another interesting expression which arises from this analysis is the general expression for the numbers defined by:

(7a) $$U_{n,m} = 0 \qquad (m = -n+1, \cdots, -1)$$

(7b) $$U_{n,m} = 1 \qquad (m = 0)$$

(7c) $$U_{n,m} = \sum_{i=1}^{m} U_{n,m-i} \qquad (m > 0)$$

It is seen that $F_m = U_{2,m-1}$.

These numbers also arise in the analysis of polyphase merge-sorting; they represent the number of strings produced at each step of the process.

The general expression is:

$$U_{n,a(n+1)+b} = 2^{b-1} \sum_{i=0}^{a} \left\{ \binom{in+a+b}{a-i} + \binom{in+a+b-1}{a-i-1} \right\} (-1)^{a-m} (2^{n+1})^{n}$$

$$(b = 0, \cdots, n), \quad (a + b > 0)$$

The proof is similar to the above and is omitted.

ACKNOWLEDGEMENT

The author is indebted to Donald E. Knuth for suggesting the proof used above.

REFERENCES

1. E. P. Miles, Jr., "Generalized Fibonacci Numbers and Associated Matrices," Amer. Math. Month. 67 (October 1960), pp 745—752.
2. R. L. Gilstad, "Polyphase Merge Sorting—an Advanced Technique," Proc. Eastern Joint Computer Conference, Dec. 1960.
3. G. Polya, Induction and Analogy in Mathematics, Princeton, Chapter, 5, p. 77.

* * * * *

FIBONACCI YET AGAIN

J. A. H. Hunter

Consider a triangle such that the square of one side equals the product of the other two sides.

Then we have sides: $X, \sqrt{XY}$, and Y; say $X > Y$.

Eliminating an common factor we may set $X = a^2$, $Y = b^2$, so that the "reduced" sides become a^2, ab, b^2.

Then, for a triangle, we must have $ab + b^2 > a^2$ which requires $(\sqrt{5} - 1)/2 < b/a < (\sqrt{5} + 1)/2$.

Hence a sufficient condition for a triangle that meets the requirements is

$$F_{2n-1}/F_{2n} < b/a < F_{2n}/F_{2n-1} \quad \text{with} \quad X = ka^2, \; Y = kb^2 \; .$$

* * * * *

All subscription correspondence should be addressed to Brother U. Alfred, St. Mary's College, Calif. All checks ($4.00 per year) should be made out to the Fibonacci Association or the Fibonacci Quarterly. Manuscripts intended for publication in the Quarterly should be sent to Verner E. Hoggatt, Jr., Mathematics Department, San Jose State College, San Jose, Calif. All manuscripts should be typed, double-spaced. Drawings should be made the same size as they will appear in the Quarterly, and should be done in India ink on either vellum or bond paper. Authors should keep a copy of the manuscripts sent to the editors.

* * * * *